Improved lower bounds for $\mathbf{QAC}^0$

Malvika Raj Joshi
UC Berkeley

Based on joint work with:

Avishay Tal
UC Berkeley

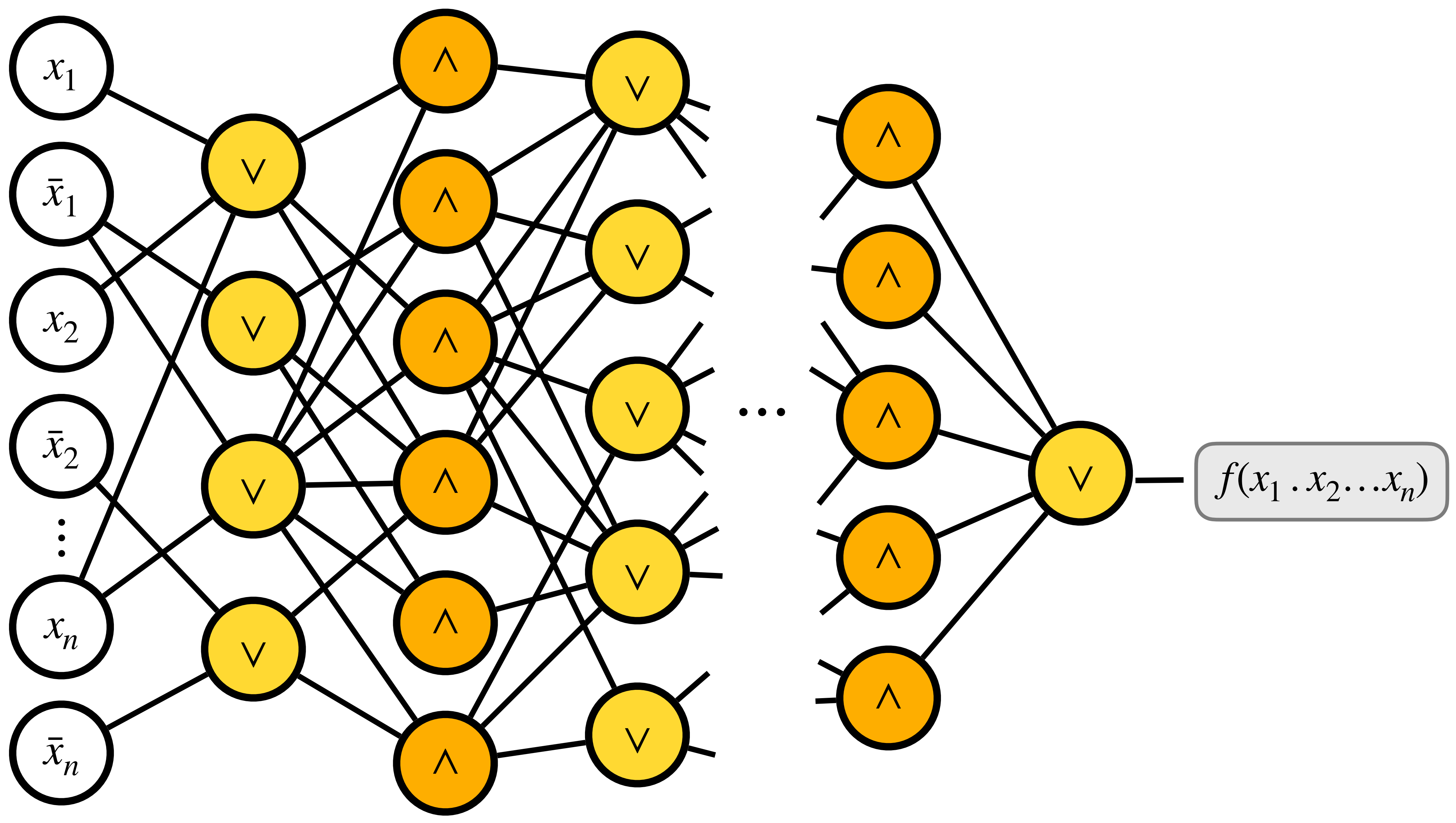
Francisca Vasconcelos
UC Berkeley

John Wright
UC Berkeley

STOC 2026

$O(1)$ depth, $\mathrm{poly}(n)$ size circuits

$\mathbf{AC^0}$: AND + NOT



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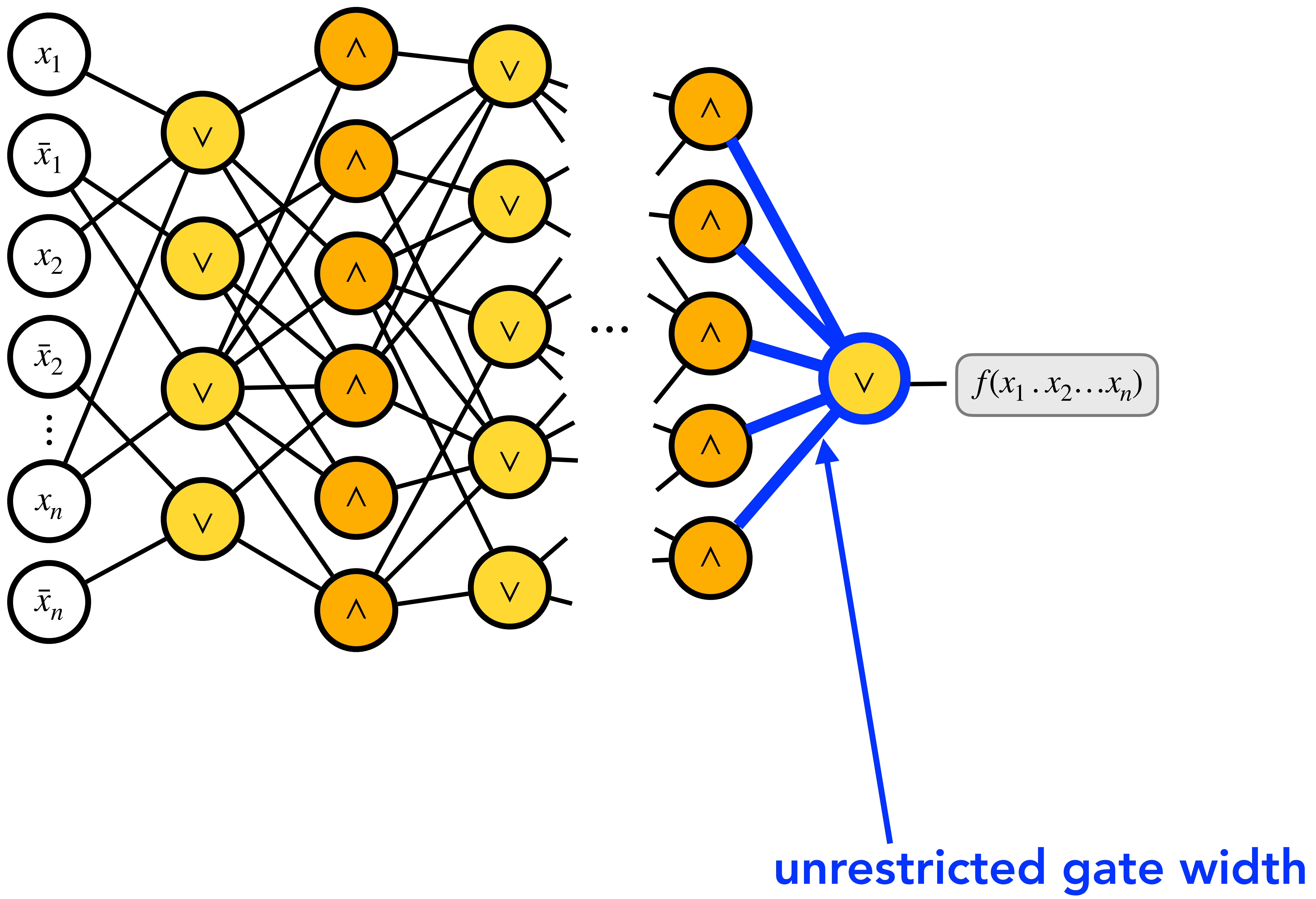

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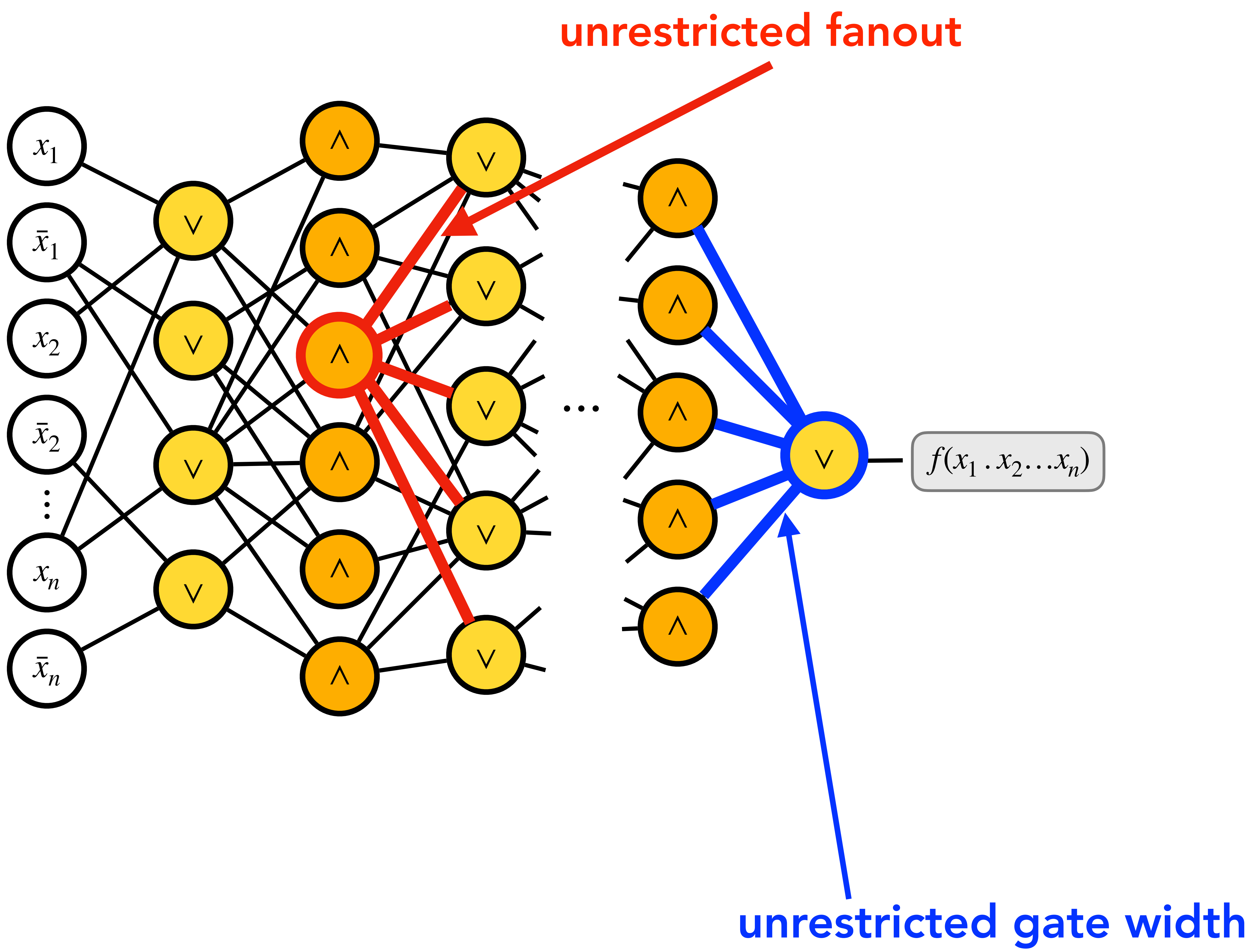

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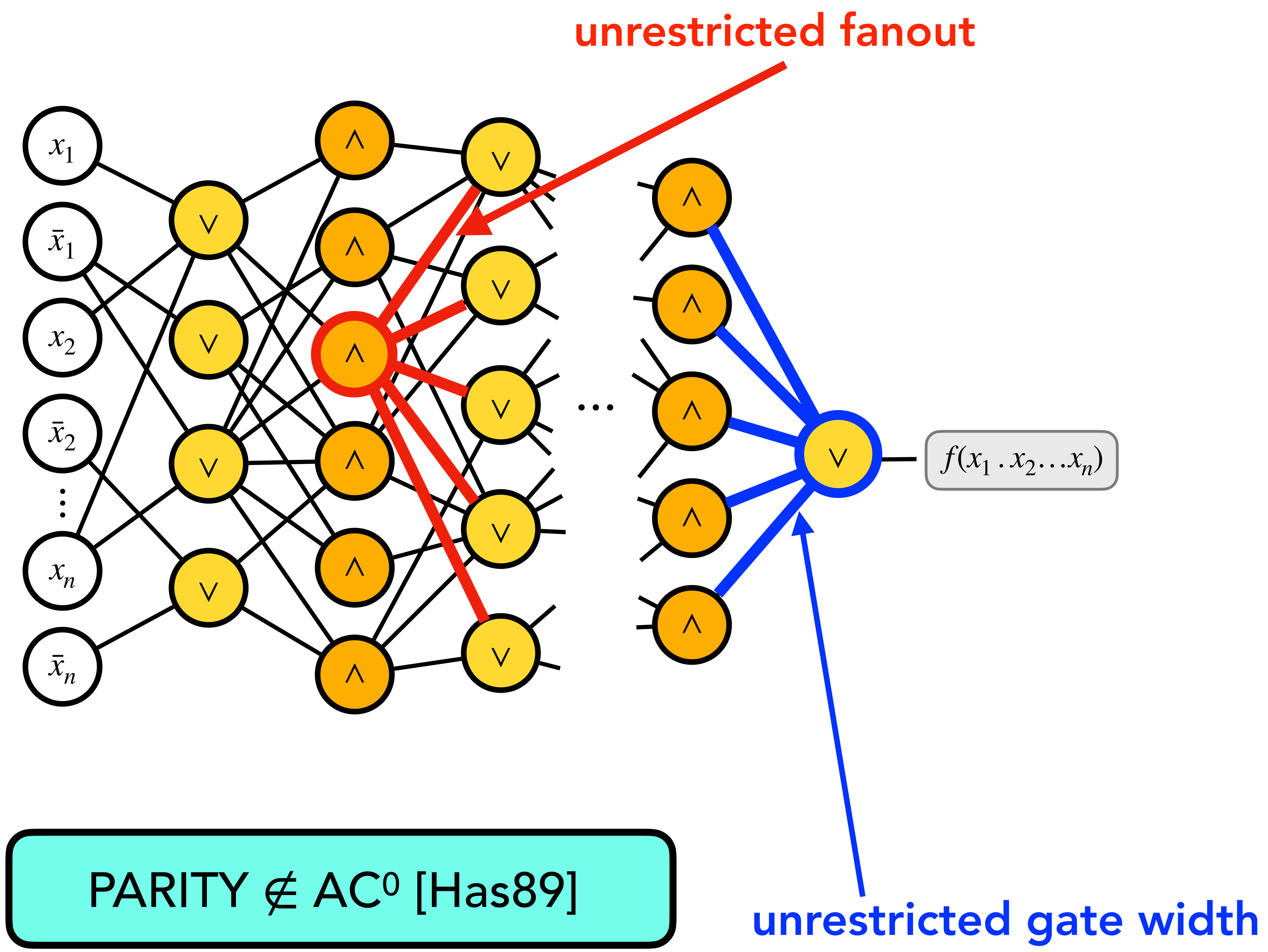

PARITY $\notin$ AC⁰ [Has89]

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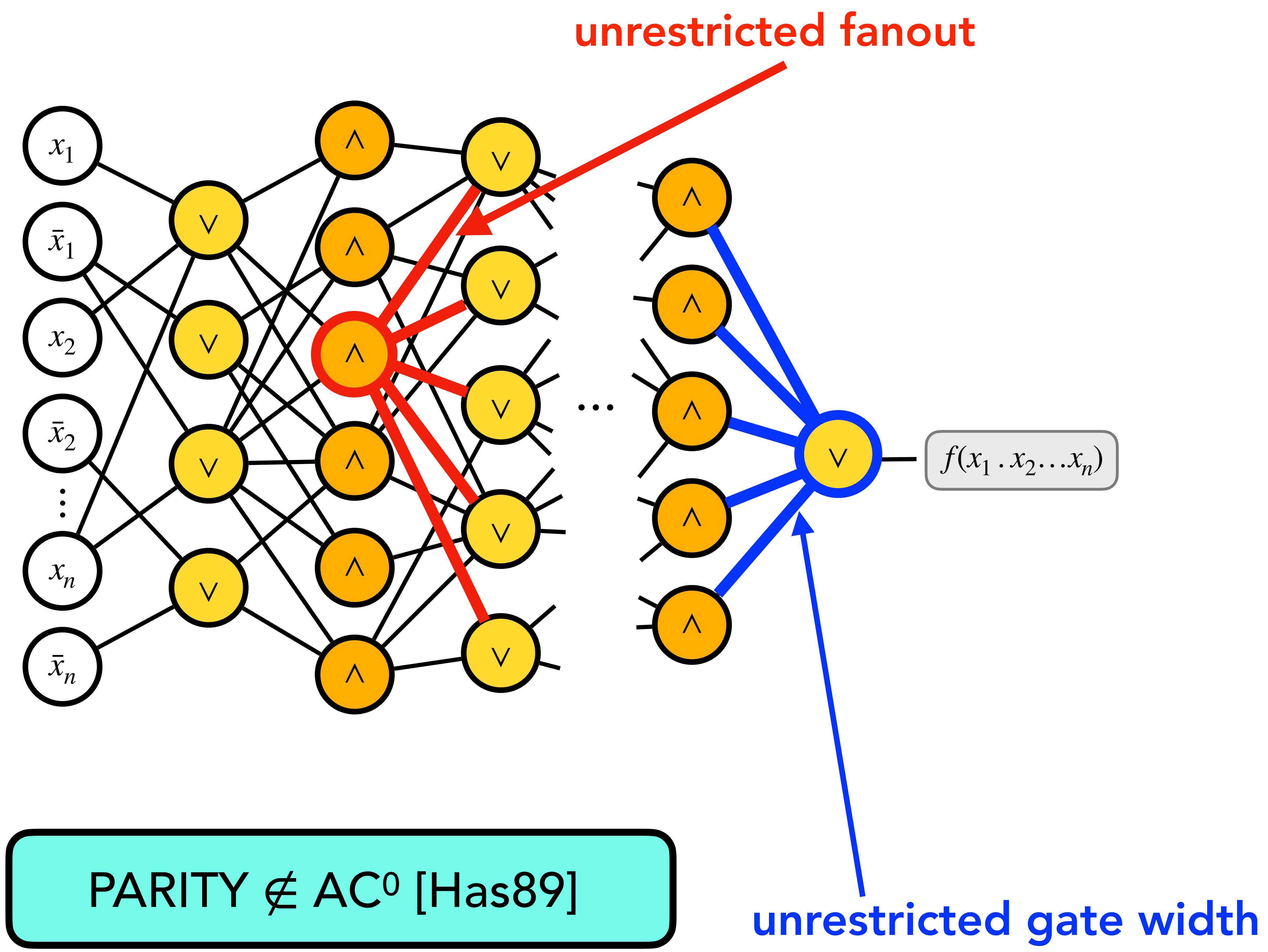
$\mathbf{QAC^0}$: Reversible-AND



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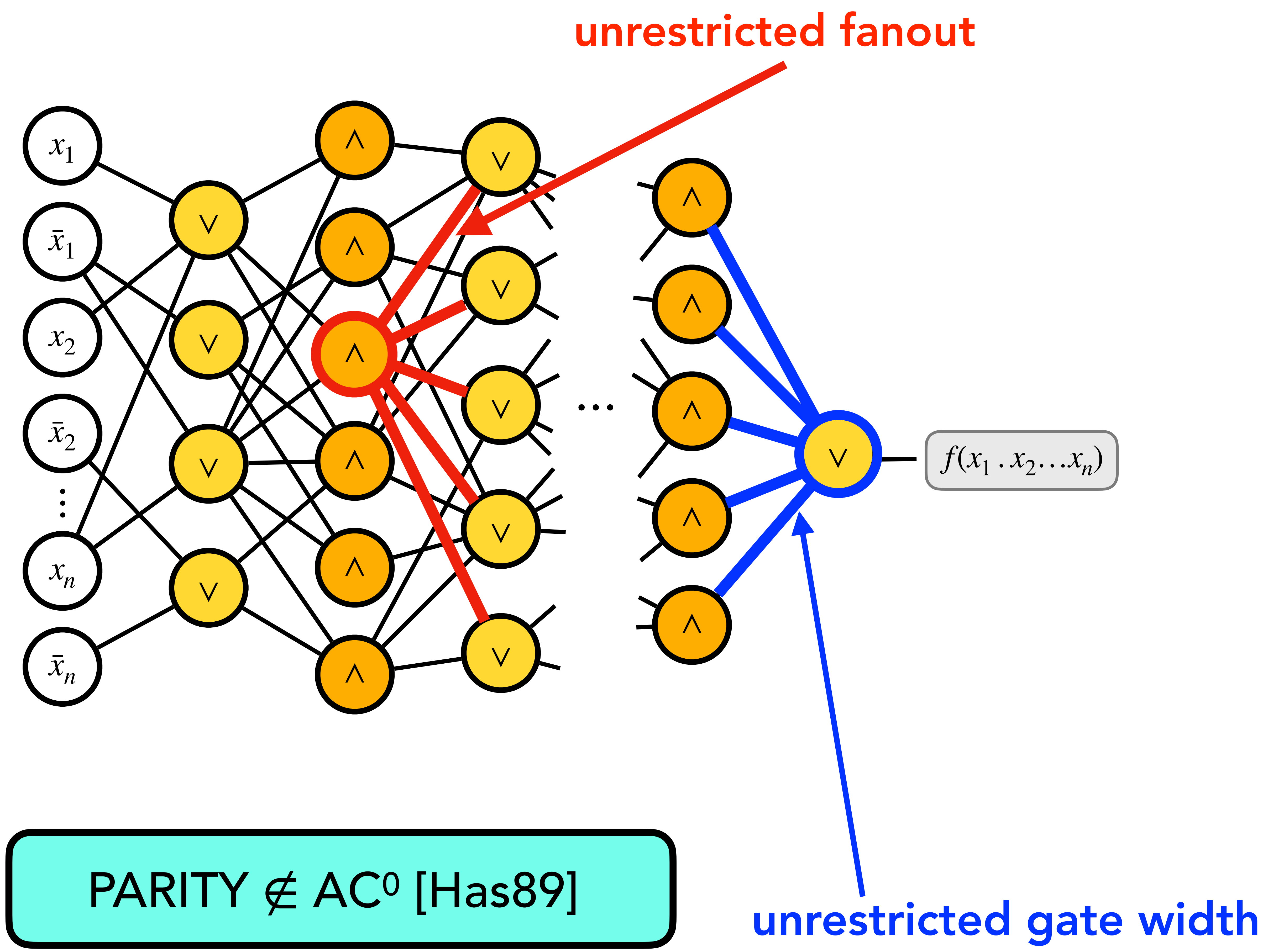

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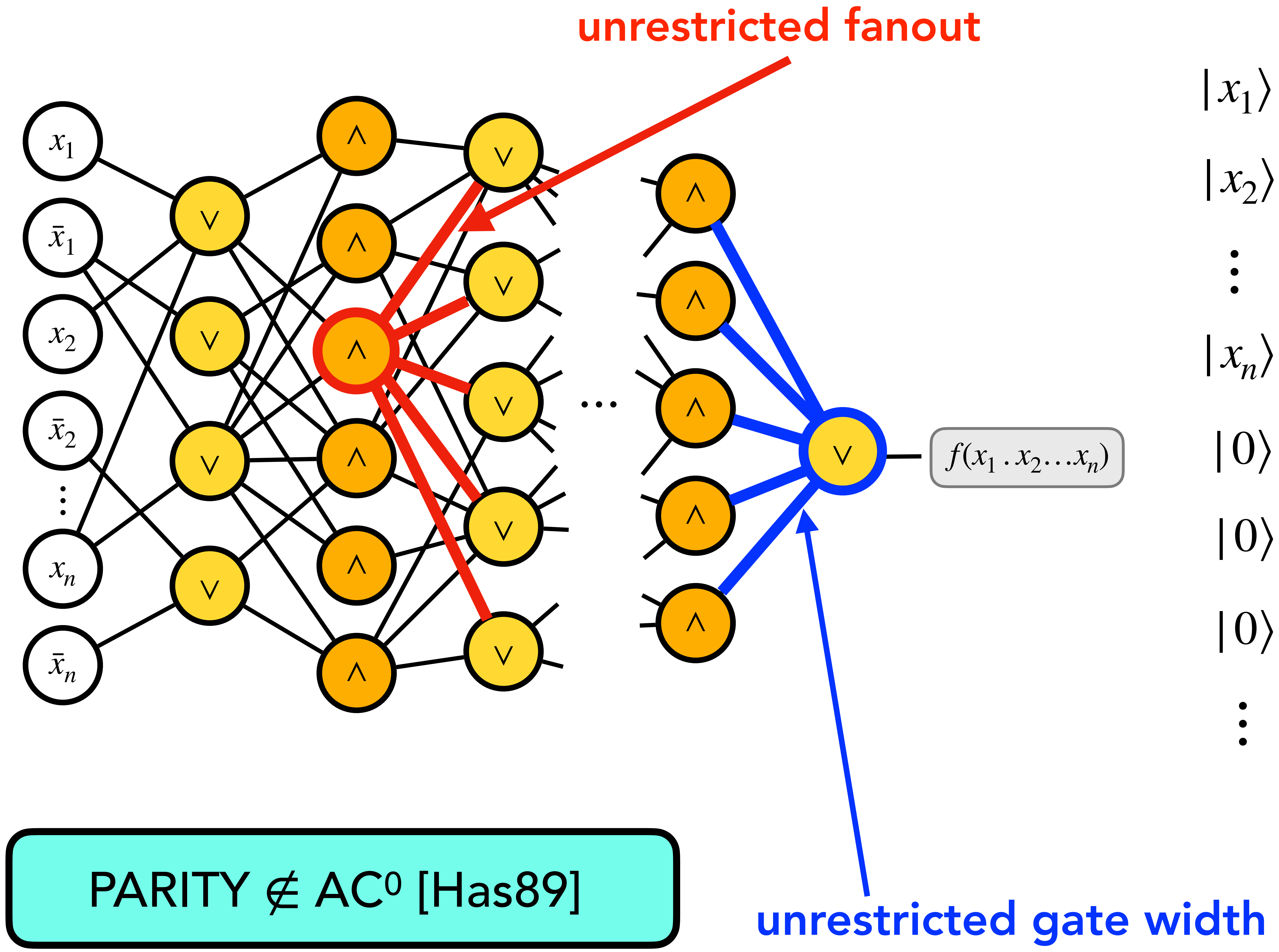

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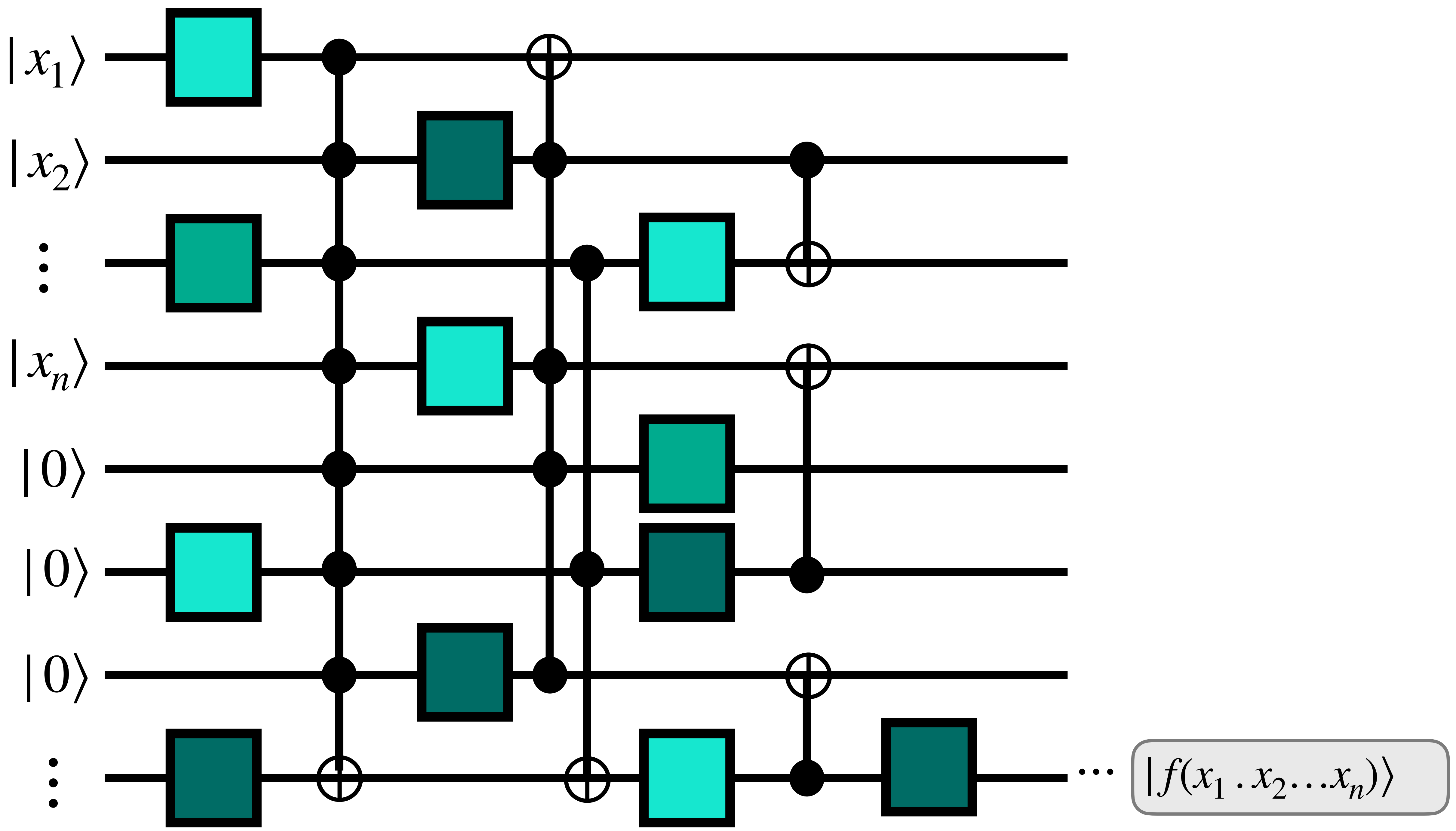

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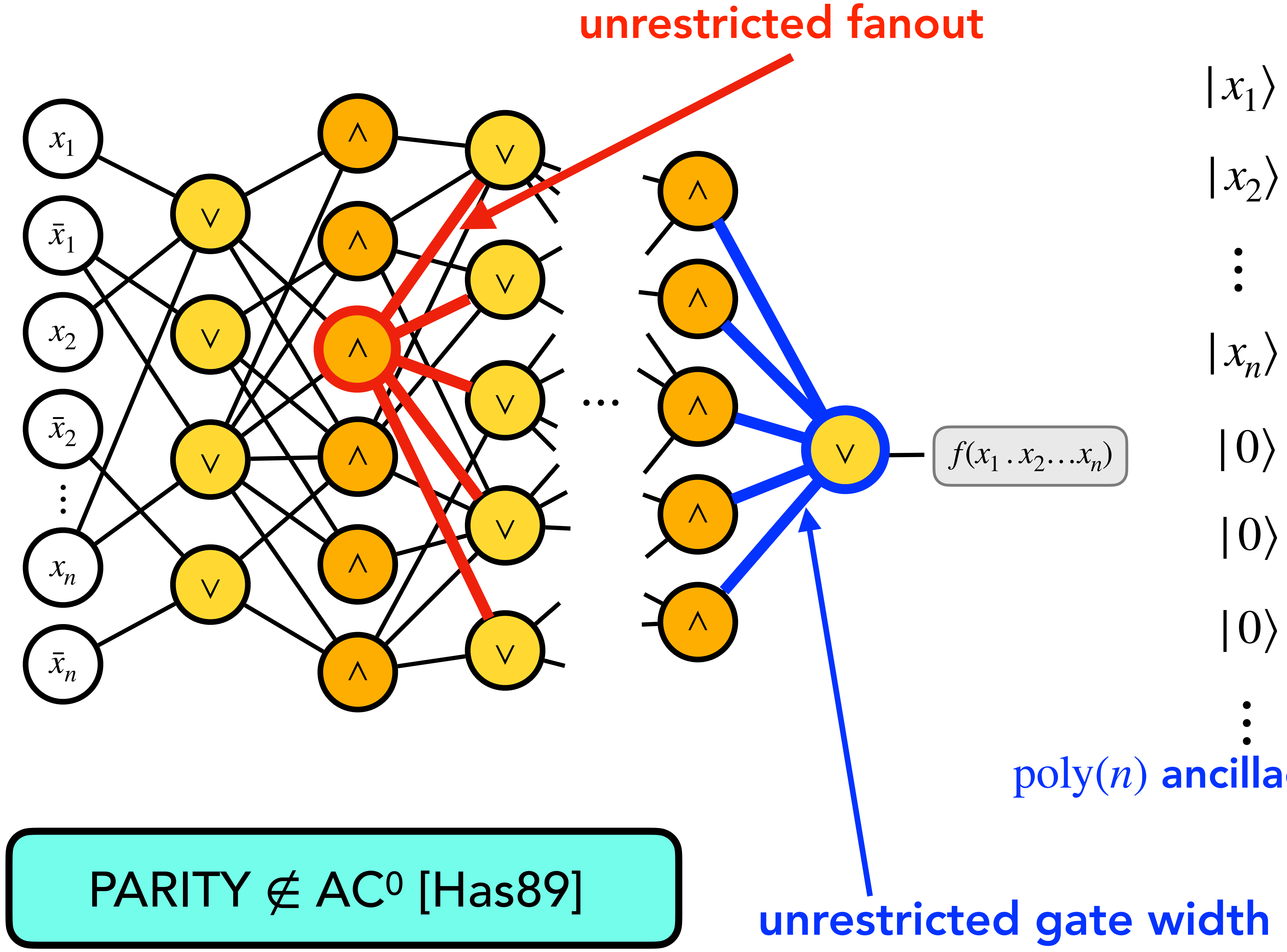

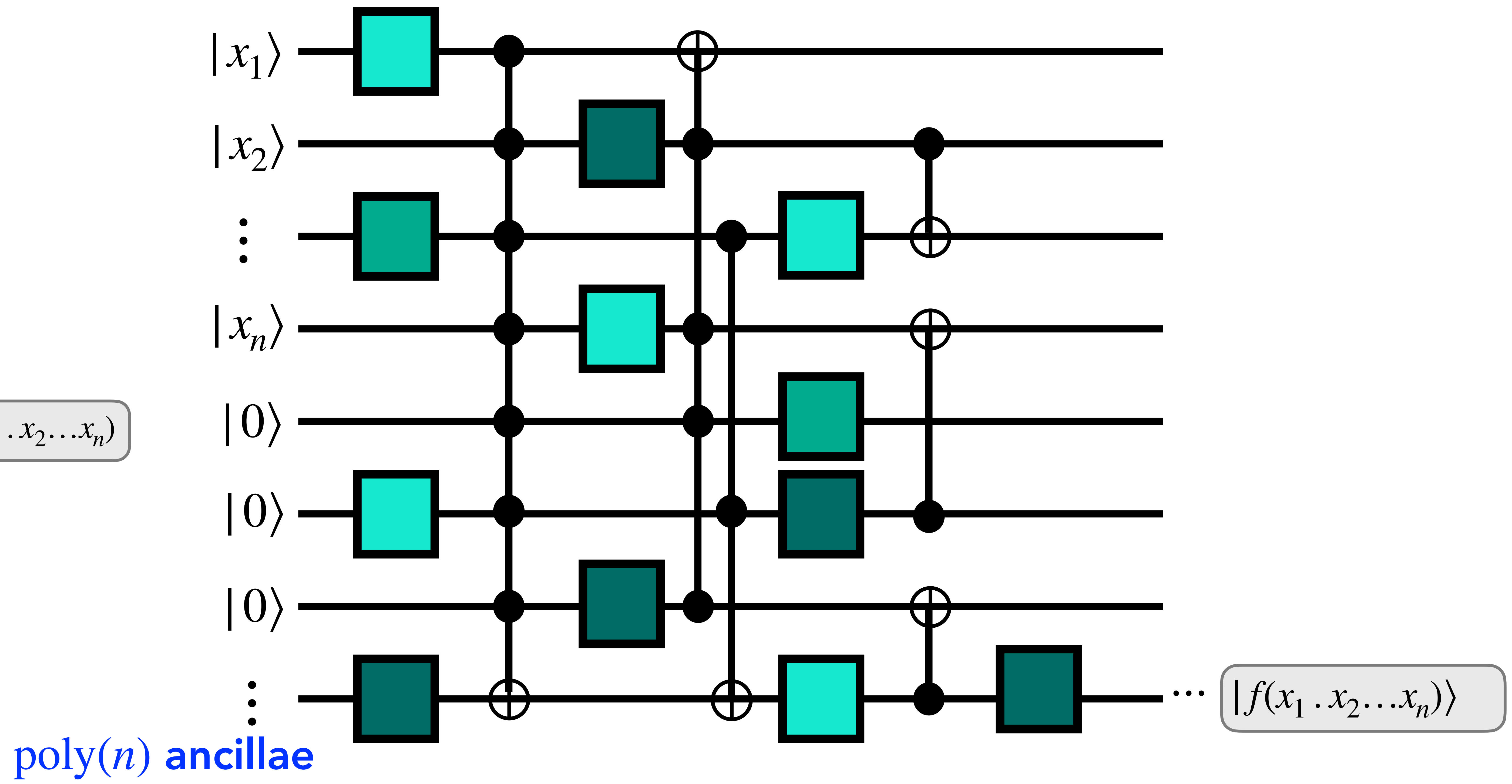

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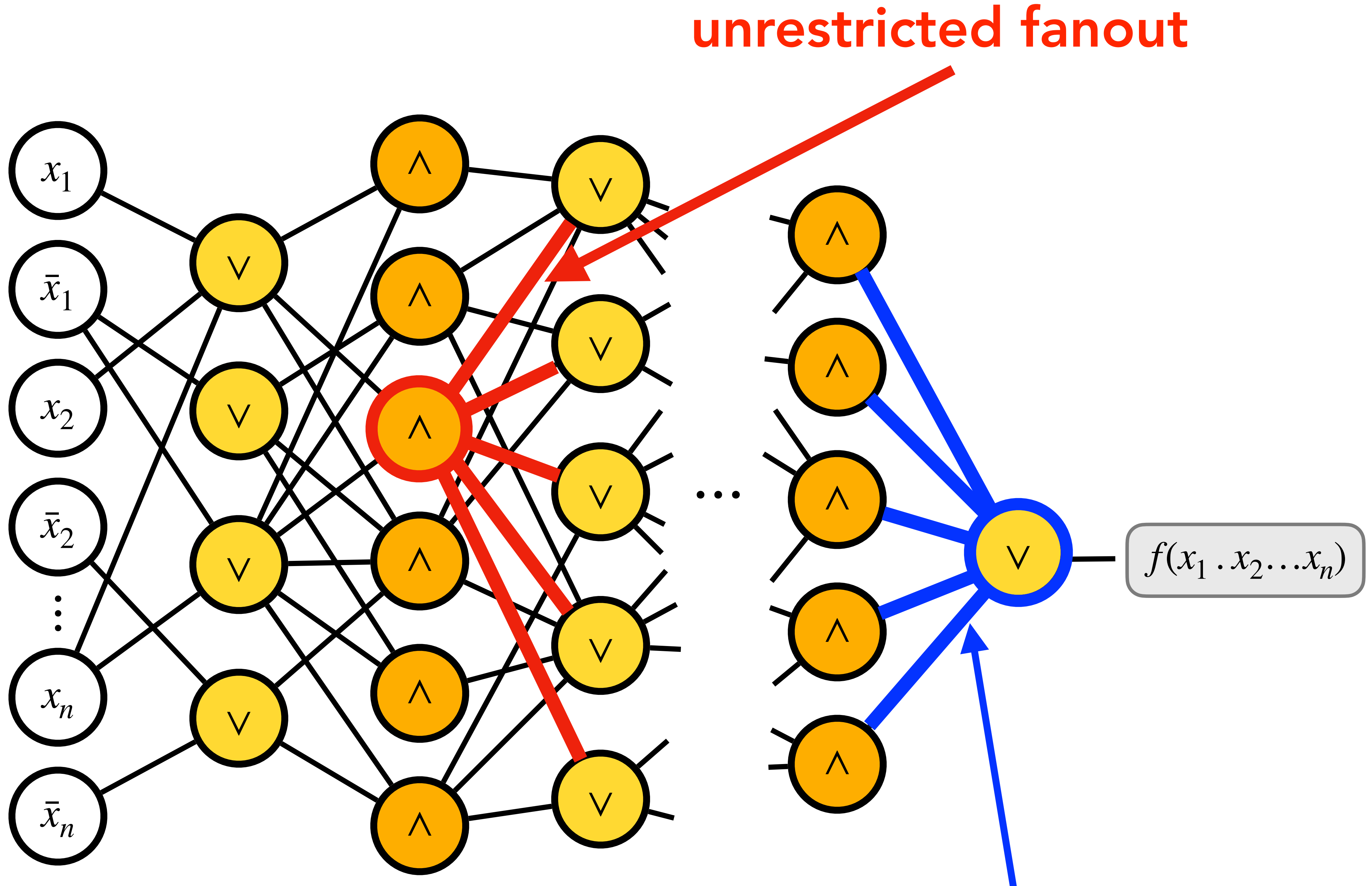

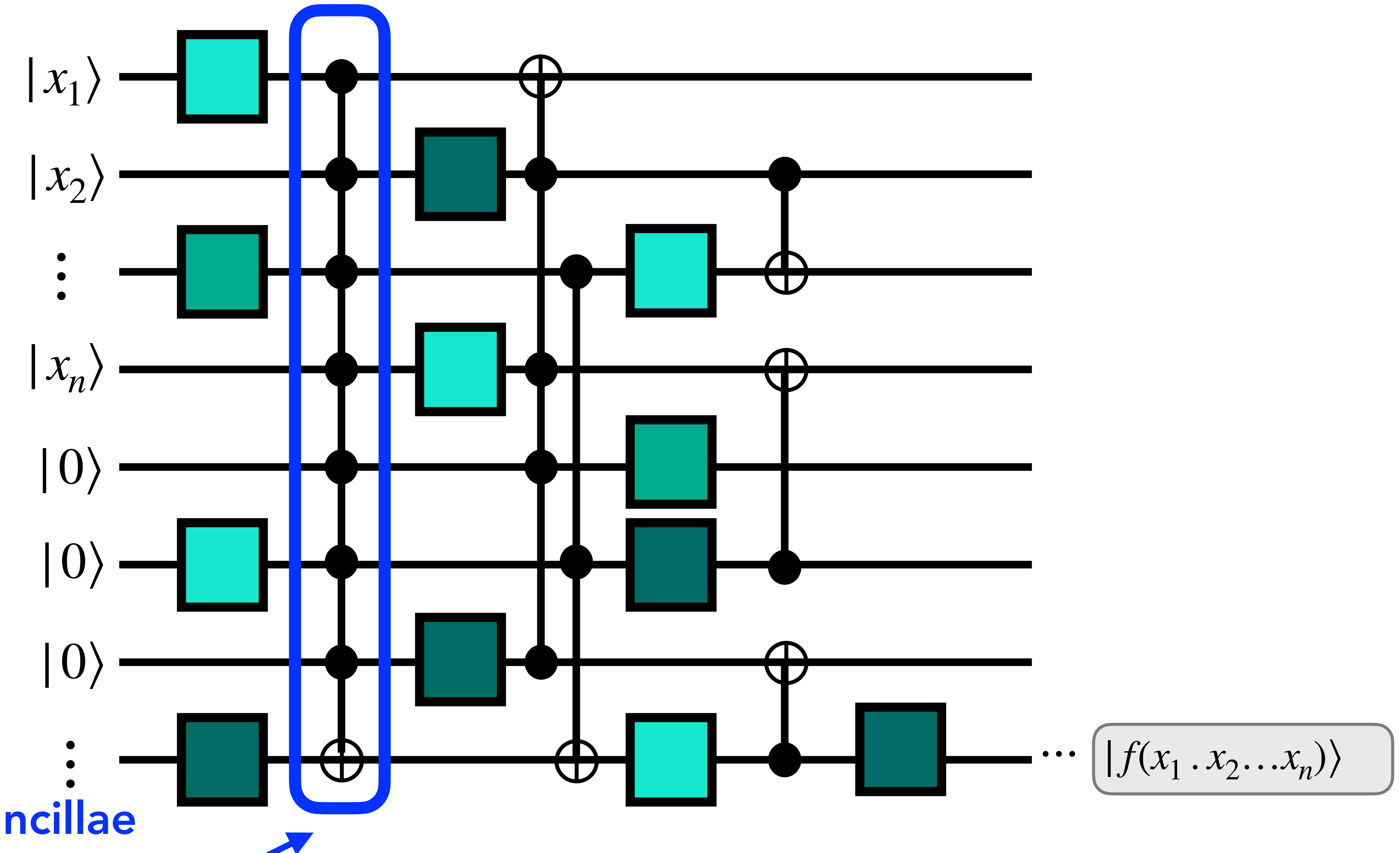

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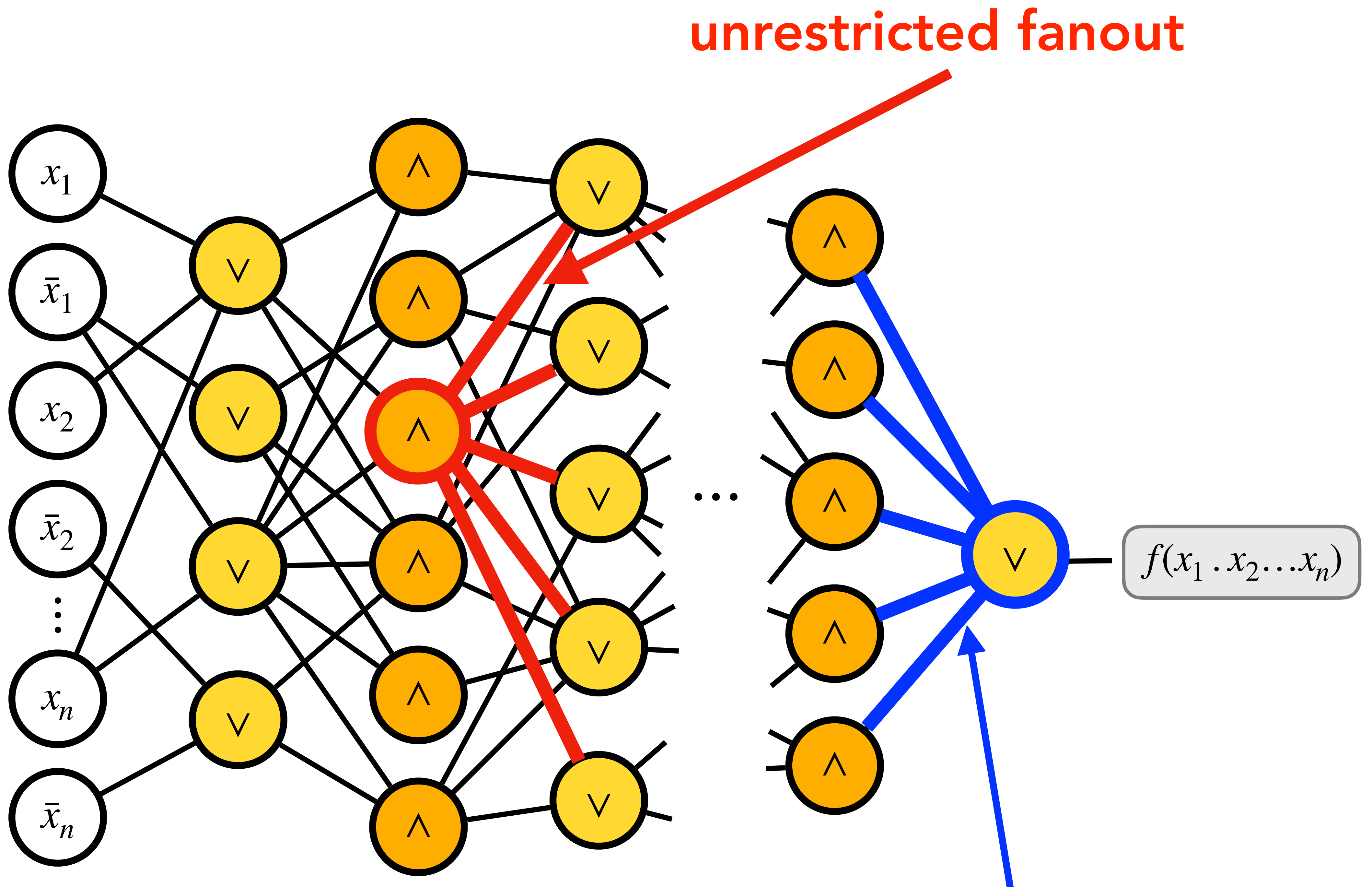

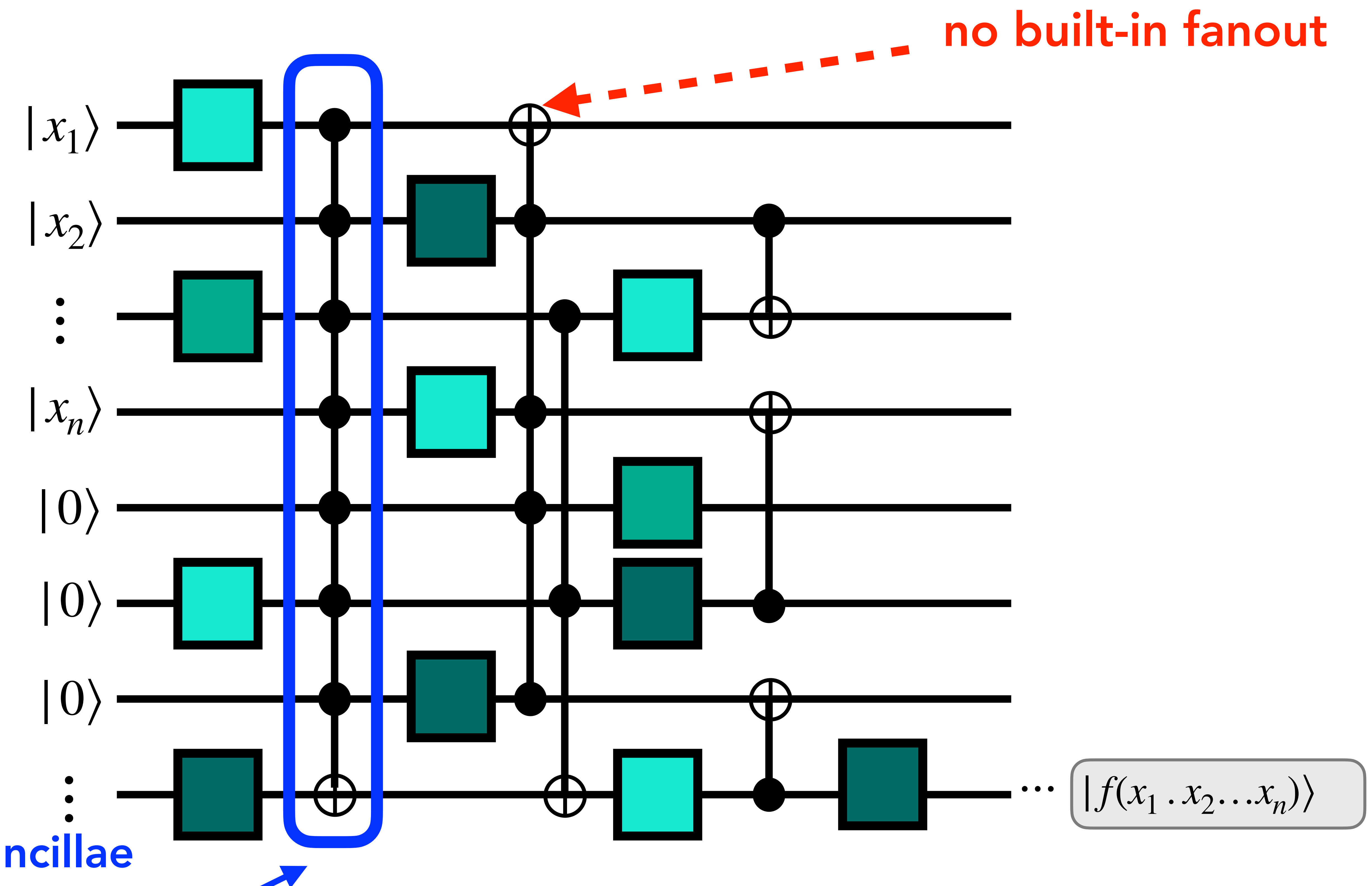

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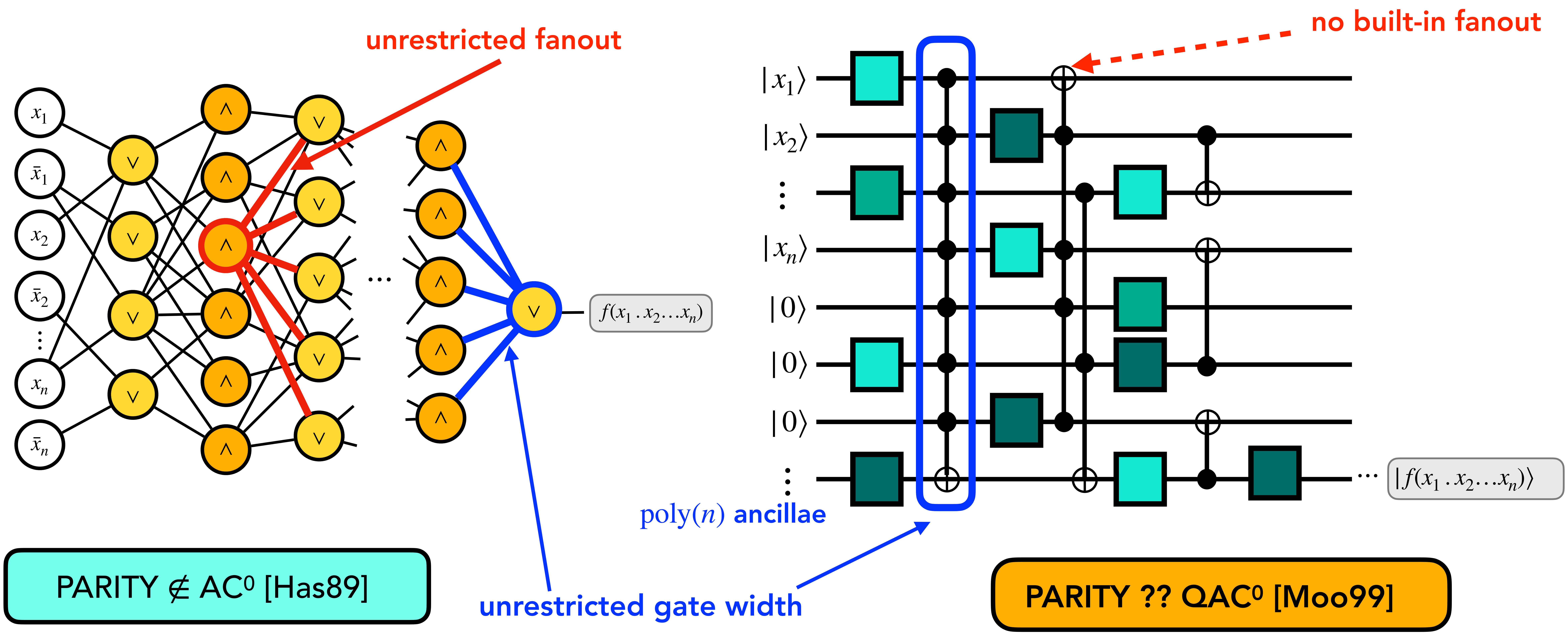

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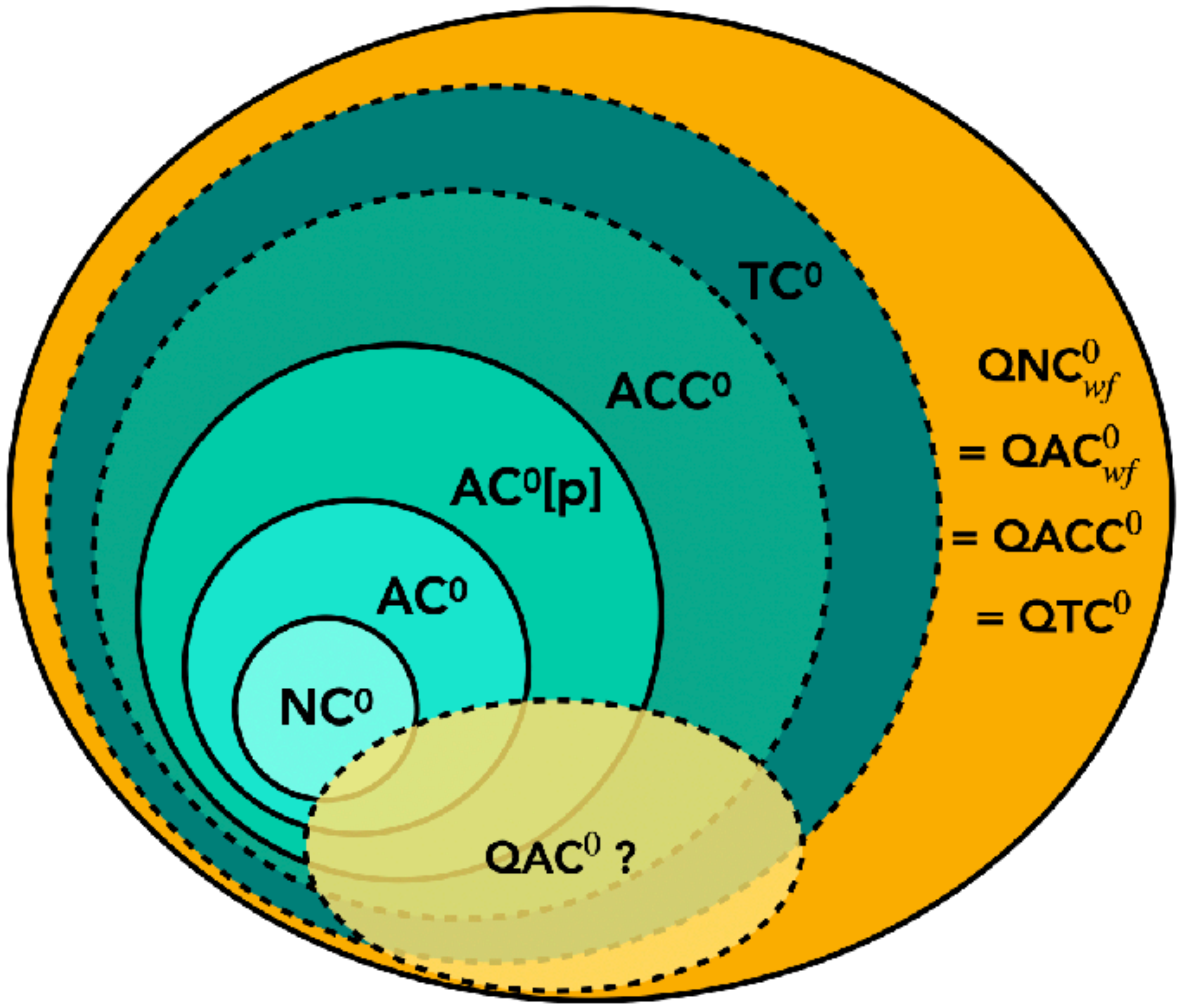
PARITY ?? QAC⁰ [Moo99]

Motivation: Why QAC^0?

If you like **complexity theory**:



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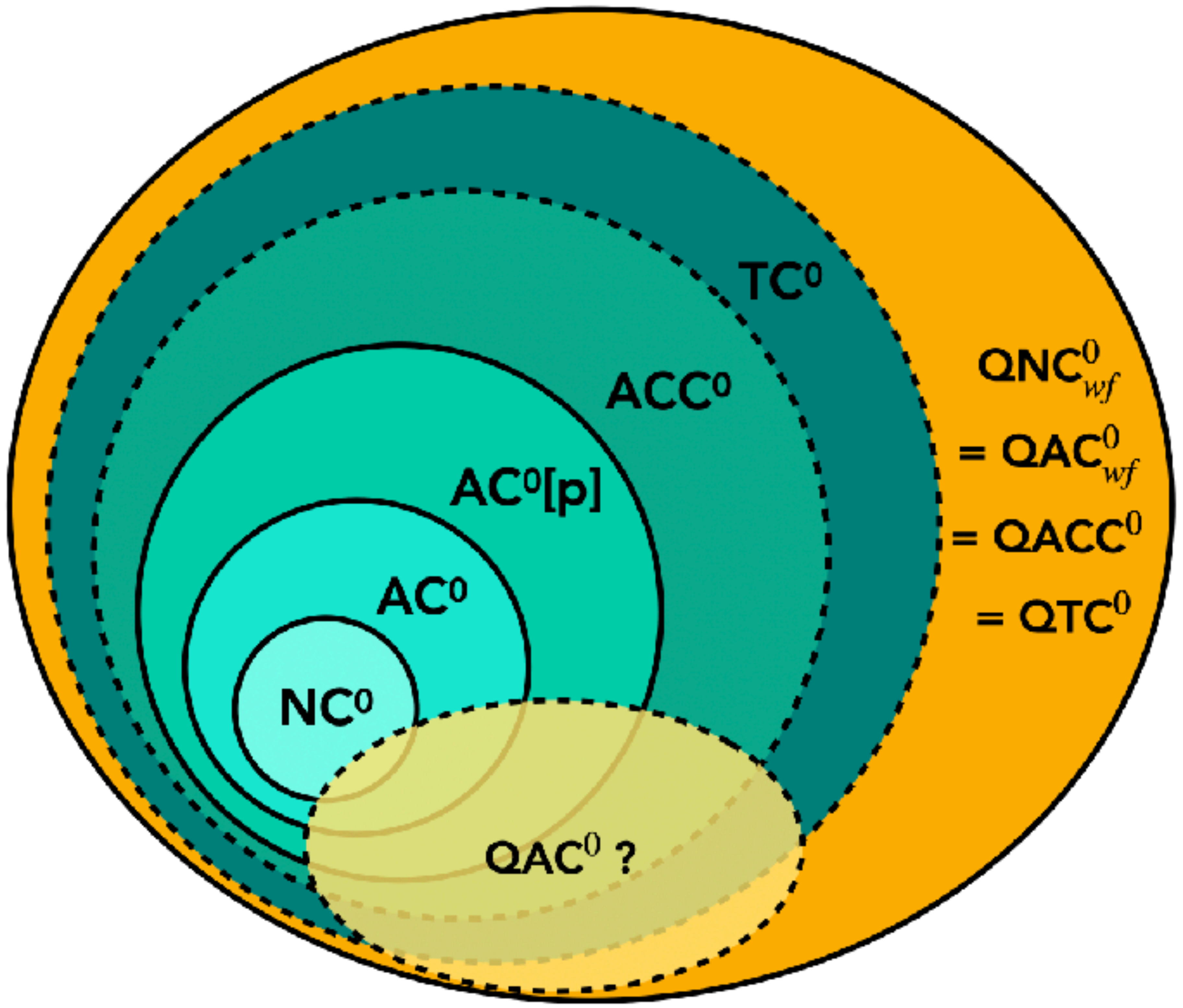

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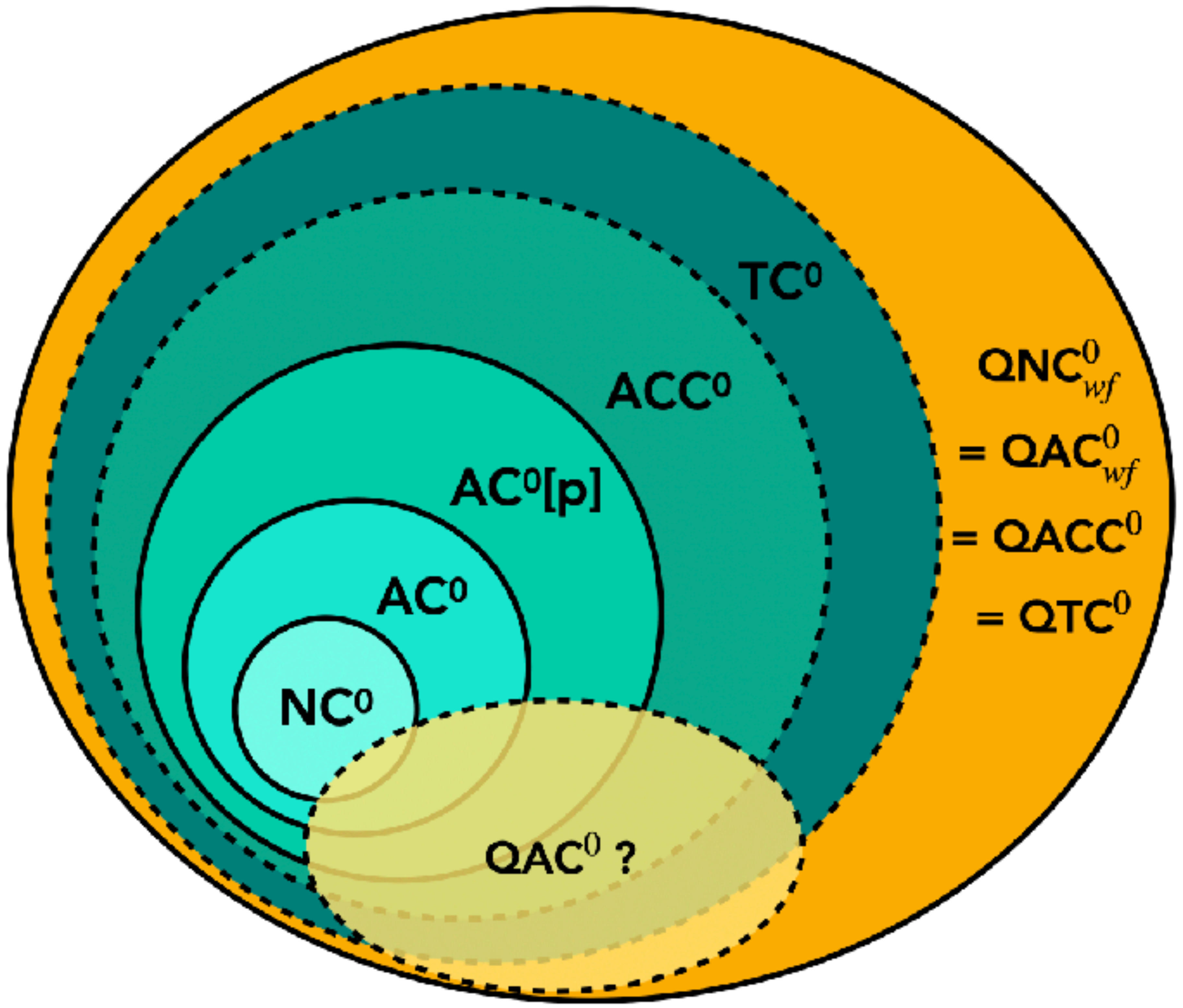

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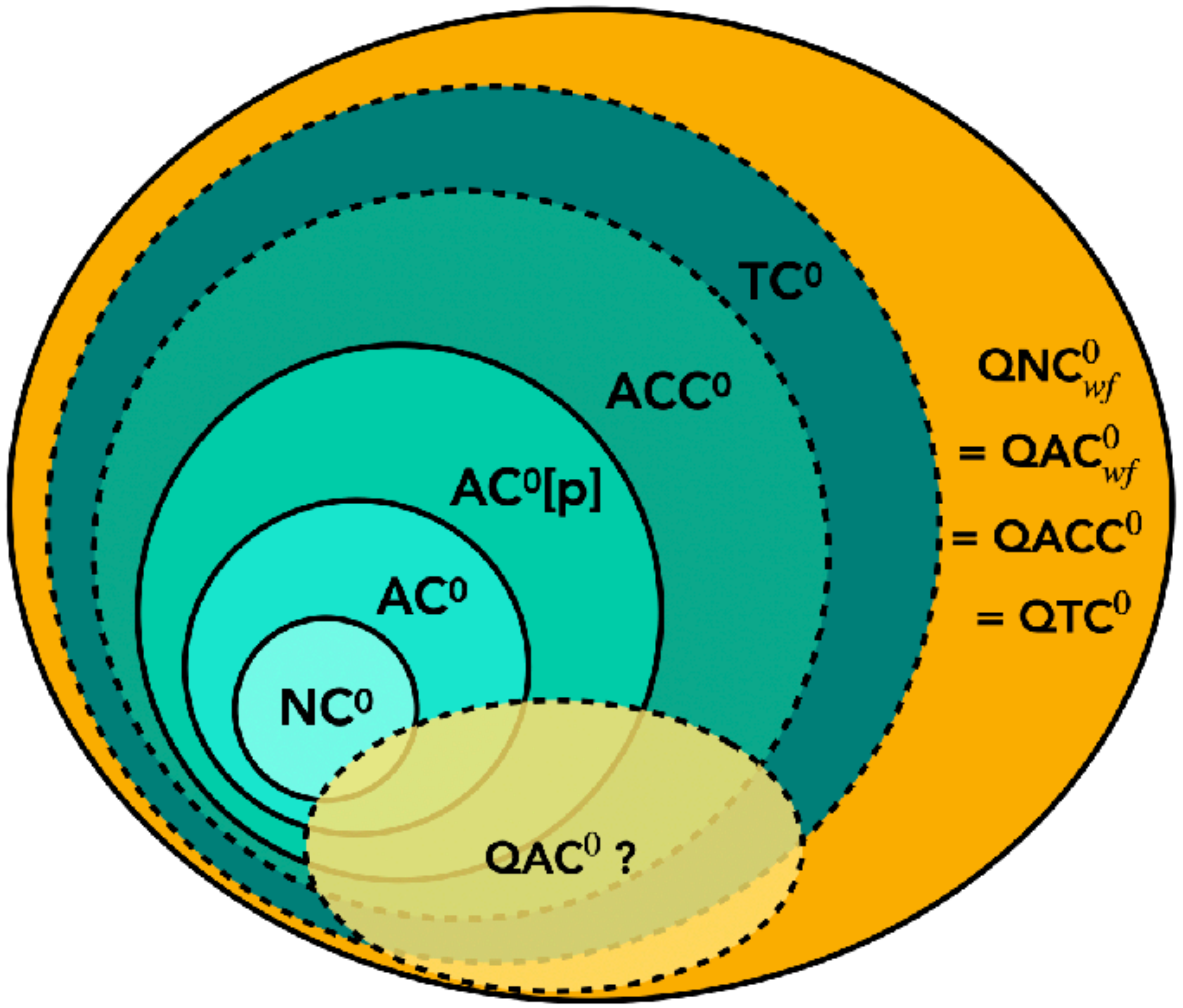

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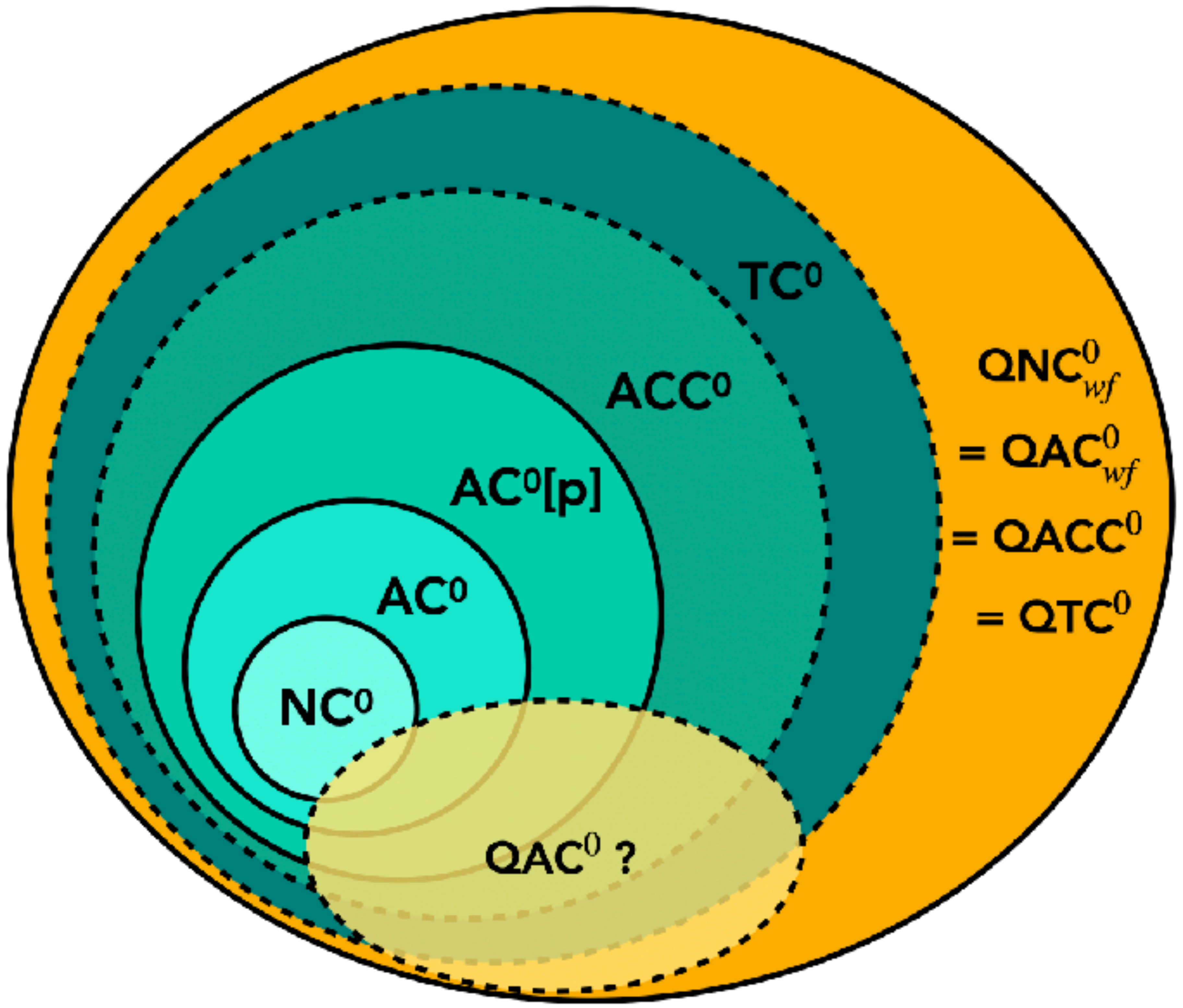

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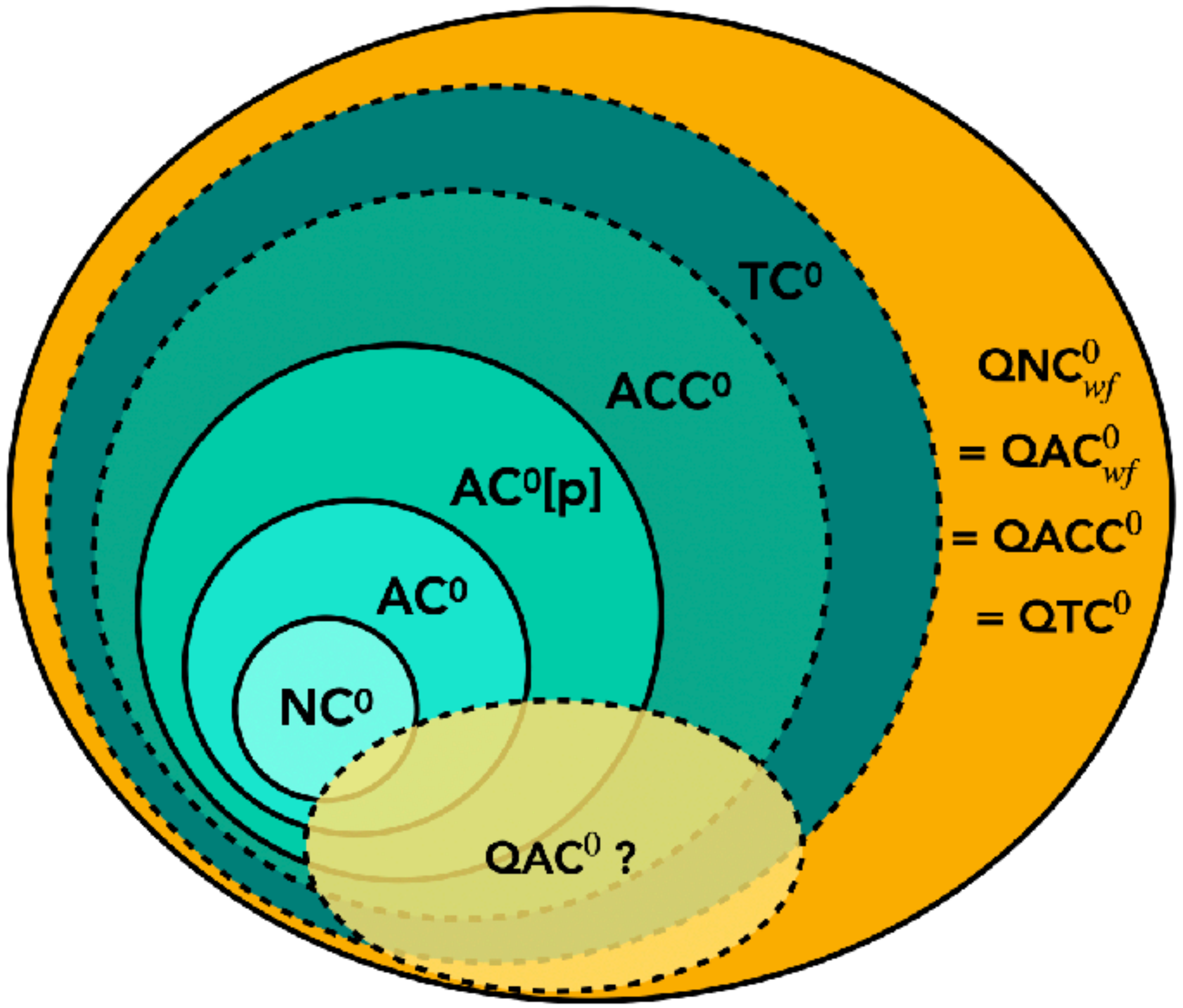

If you like **real life**:

- **Noisy Intermediate Scale Quantum Era**: practically limited to $O(1)$-depth.
- Are Quantum supremacy experiments viable?
  Rely on **Quantum Fourier Transform,** highly entangled **symmetric states.**

QAC^0 : How Powerful is the Parity Gate?

Parity := Output **qubit** of $C|x_1x_2...x_n\rangle|0^m\rangle$ is $|\mathbf{x_1} \oplus \mathbf{x_2}... \oplus \mathbf{x_n}\rangle$

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Equivalent in QAC^0 to Parity

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Classical functions $\notin \mathrm{AC}^0[2]$

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- All of **TC^0**. (AC$^0 \subsetneq$ TC^0)
- Quantum Fourier Transform
- Arbitrary symmetric states,
  arbitrary O(log n) qubit states

and more…

[Moo99, GHMP02, HS05, Ros21, GM25, **HS05, FPY25, GGJ26**]

Quantum circuit complexity of Parity

Prior work

Upper bounds

- $O(\log n)$-depth, local gates

Lower bounds

Notation : (size, depth)

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Depth 3 results

- Majority $\notin \mathrm{QAC}^0(\exp o(\sqrt{n}), 3)$ (needs **exponential size)**
- Parity $\notin \mathrm{QAC}^0(\infty, 3)$

Depth 2 results

- Any output qubit has $O(\log n)$ **total influence**
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Subsequent work [GGJ26]

Exact-Parity $\notin \mathrm{QAC}^0$ $\Rightarrow$ QLMN

$\Rightarrow$ No approx. Parity

This work: Stronger QAC^0 lower bounds

Depth 3 results

- Majority $\notin \mathrm{QAC}^0(\exp o(\sqrt{n}), 3)$ (needs **exponential size**)
- Parity $\notin \mathrm{QAC}^0(\infty, 3)$

Depth 2 results

- Any output qubit has $O(\log n)$ **total influence**
  $\implies$ no approx. Parity. (Inf[Parity] = n)
- No *exact* **nekomata** ([Ros21] : can approximate)

Subsequent work [GGJ26]

Exact-Parity $\notin \mathrm{QAC}^0$ $\Rightarrow$ QLMN

$\Rightarrow$ No approx. Parity

Exact lower bounds suffice!

Background: QAC^0 canonical form [Ros'21]

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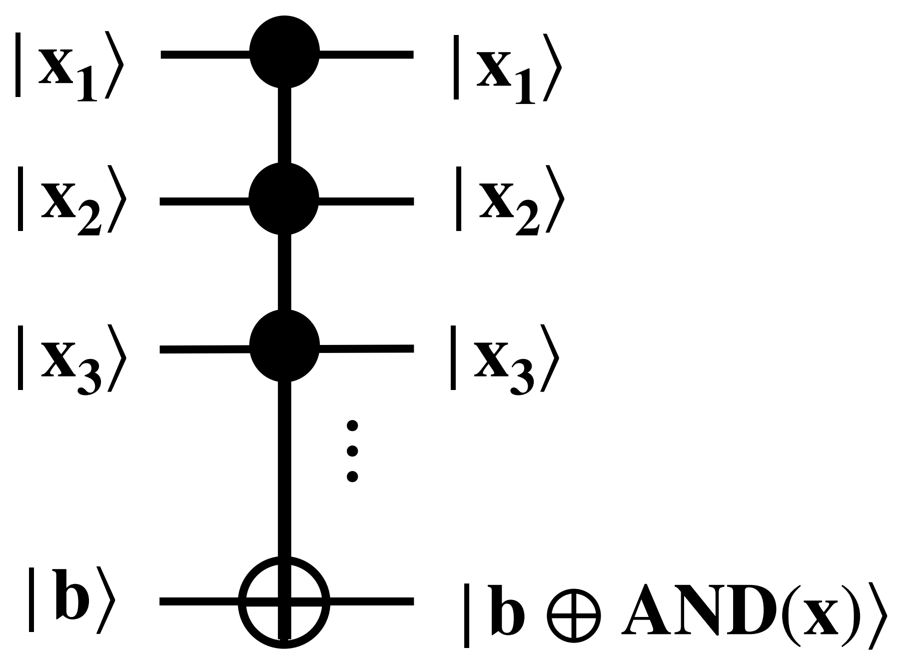
Reversible AND



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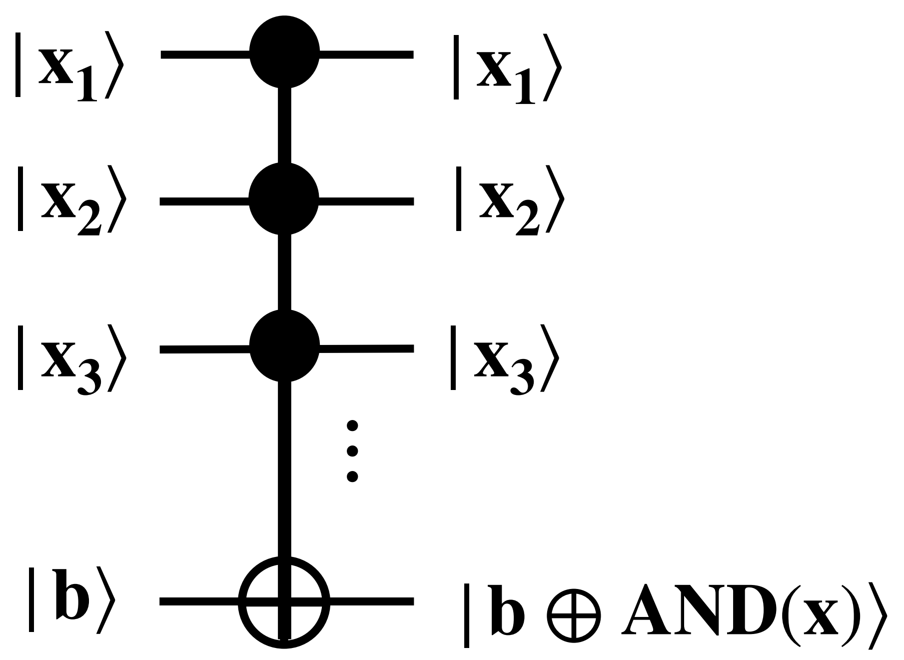

Single qubit unitaries $\longrightarrow$ no special output

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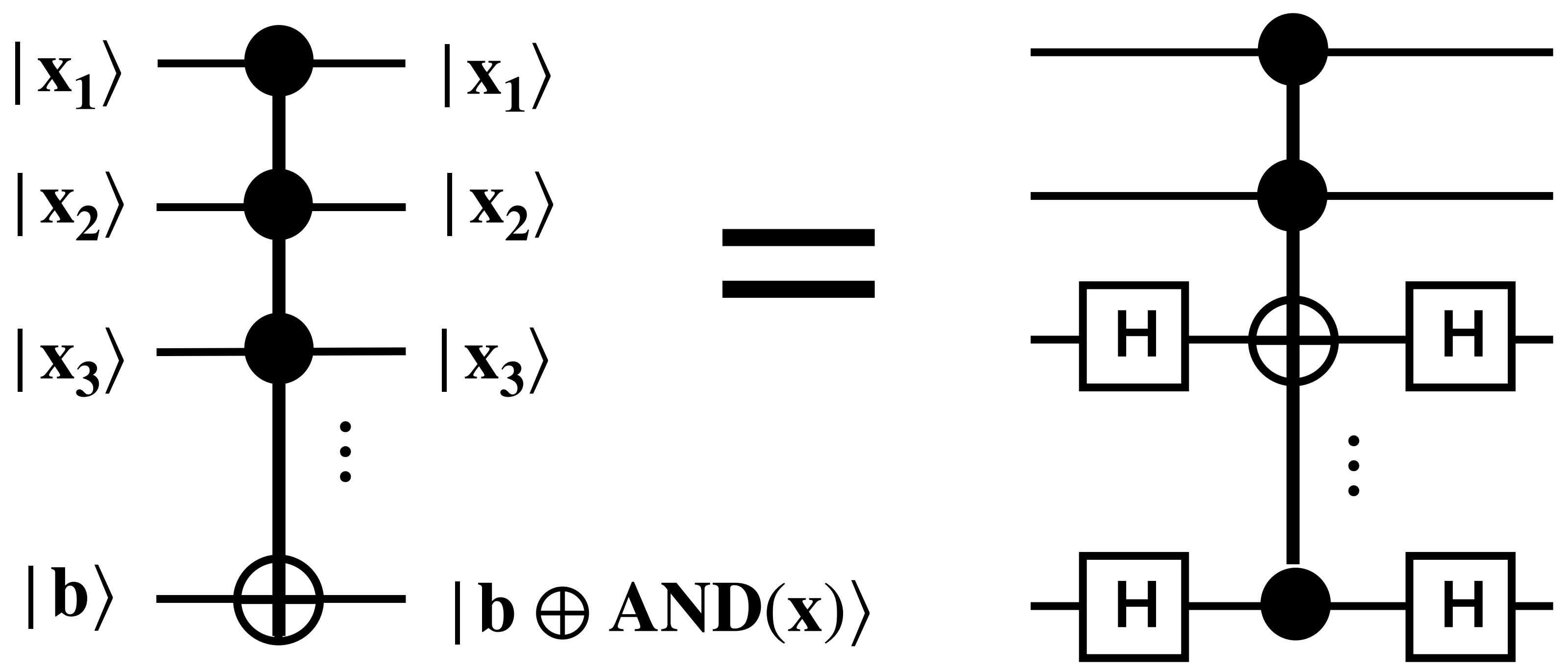

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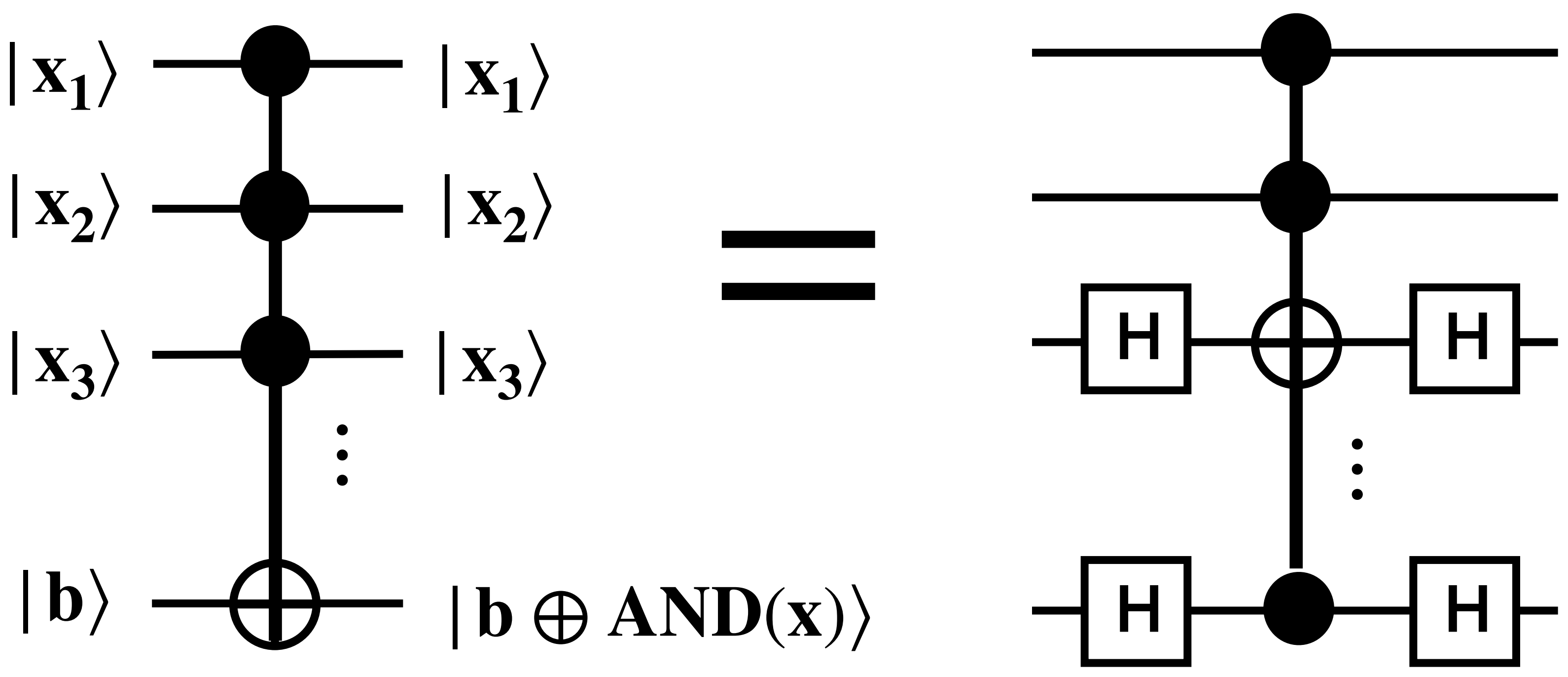

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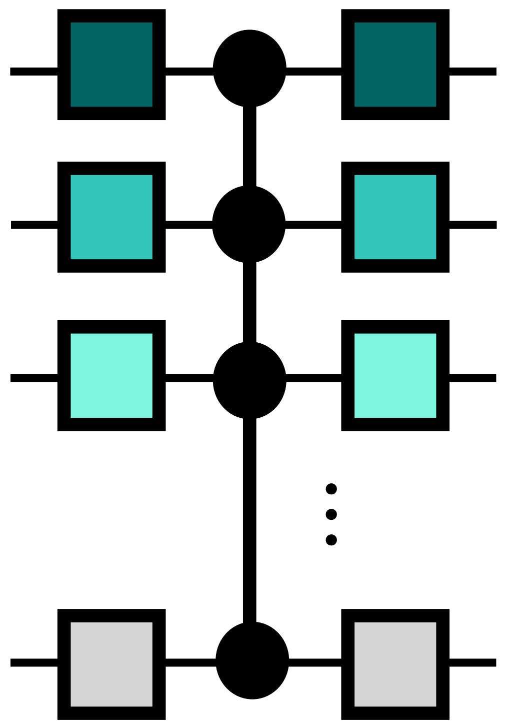
Reflections about product state



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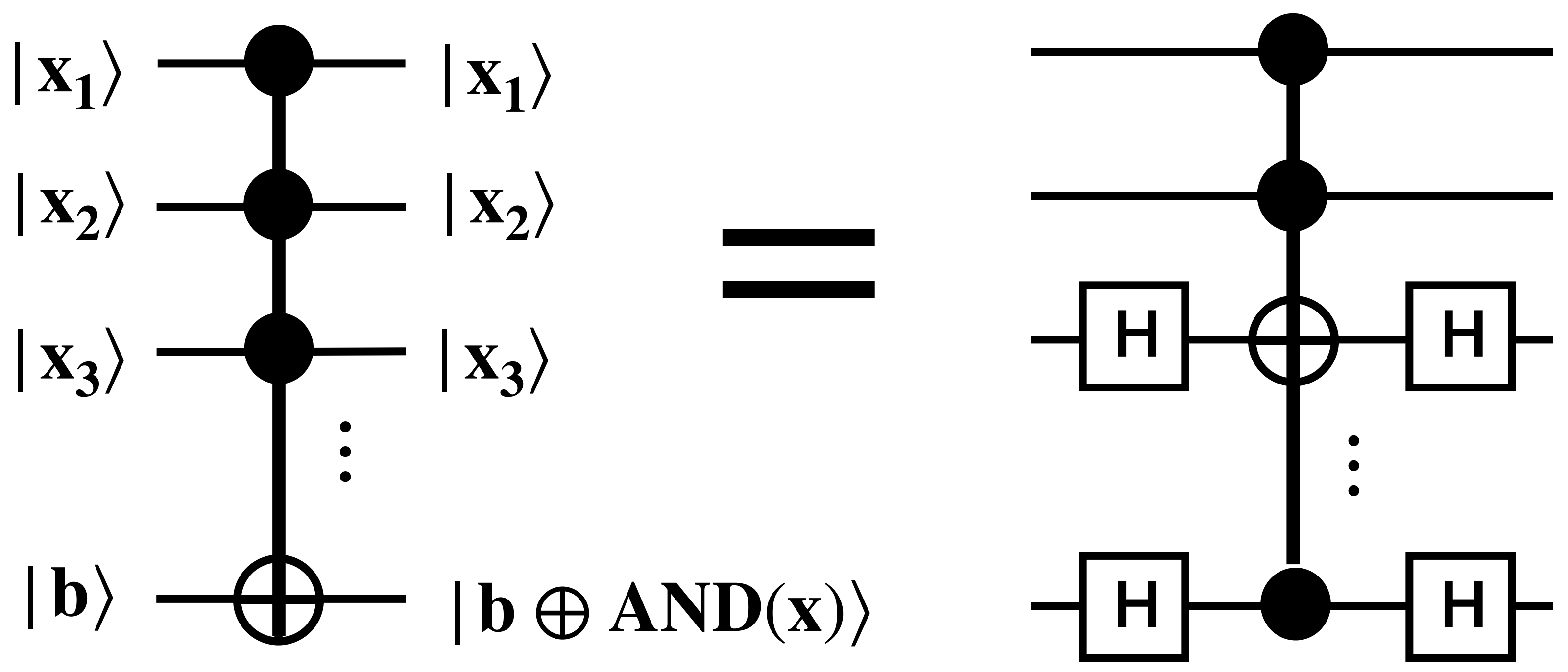

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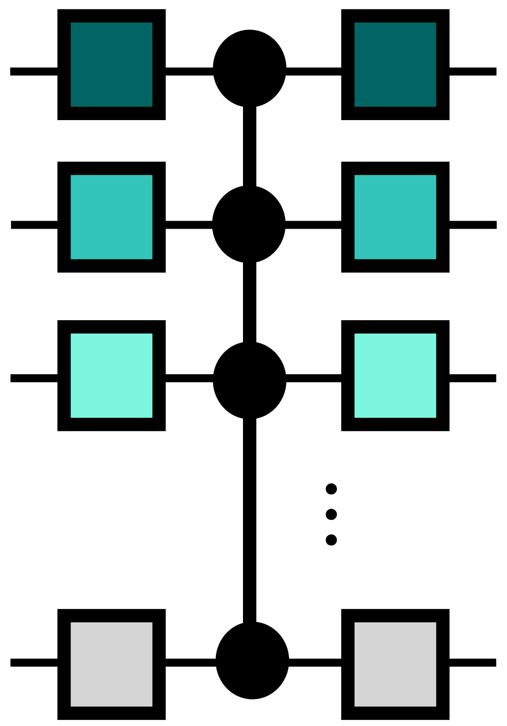

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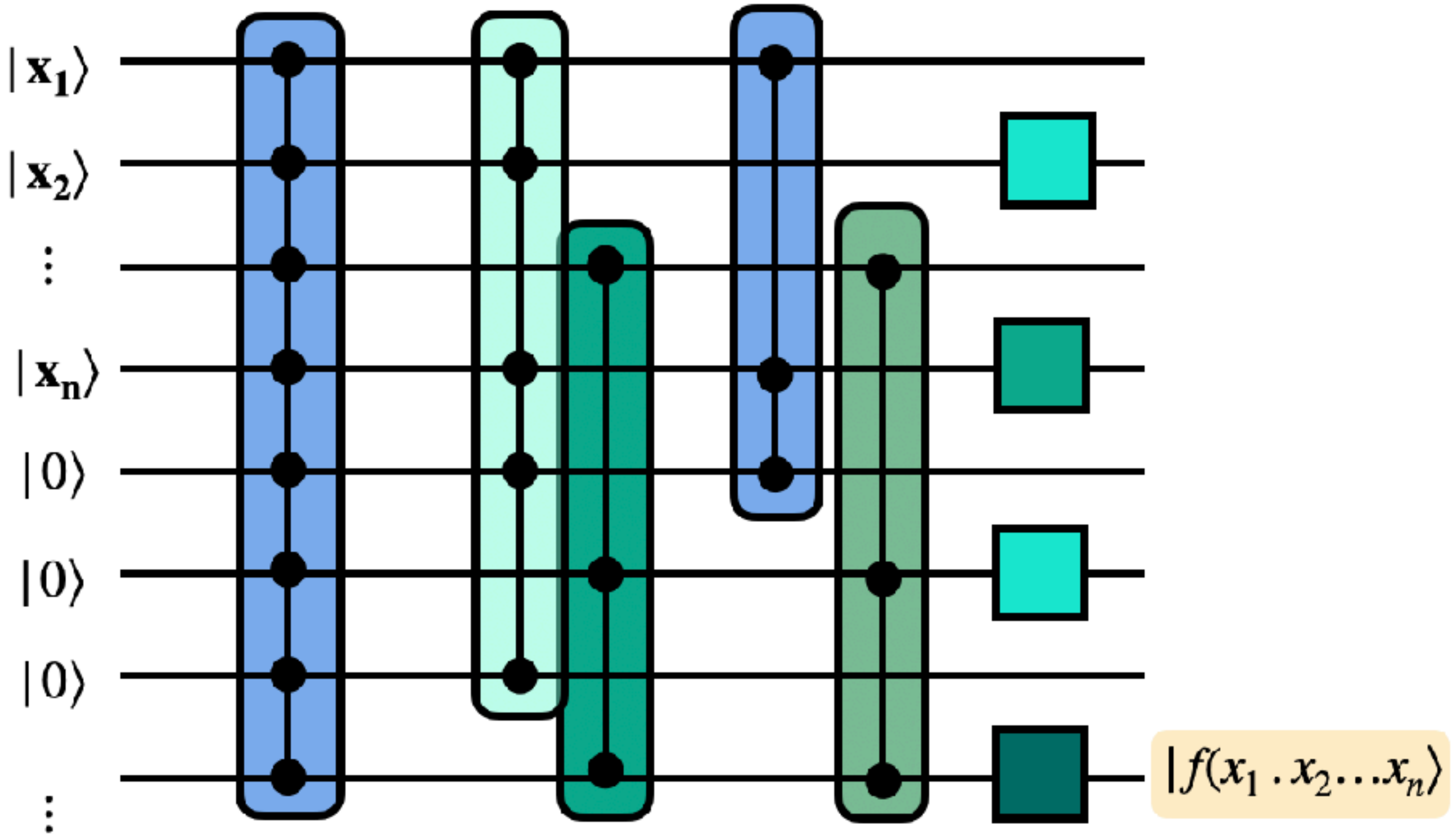

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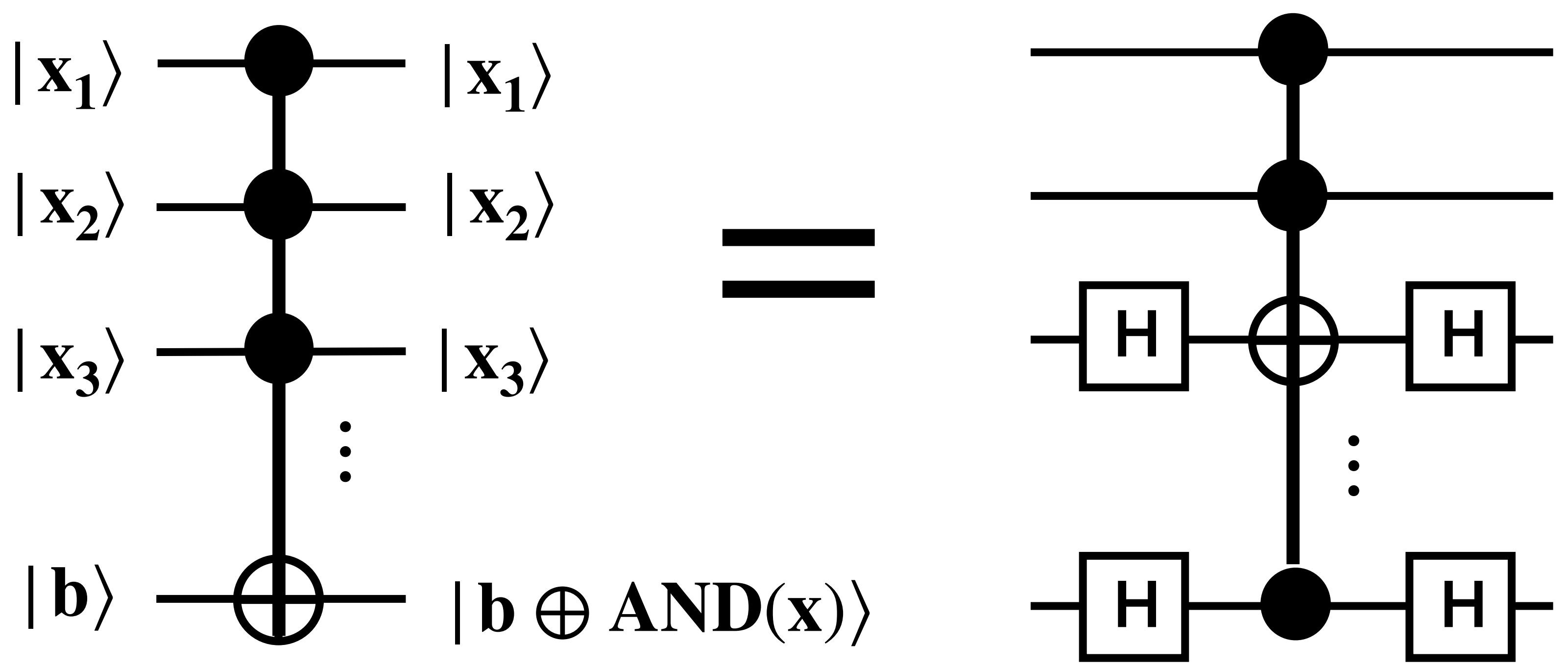

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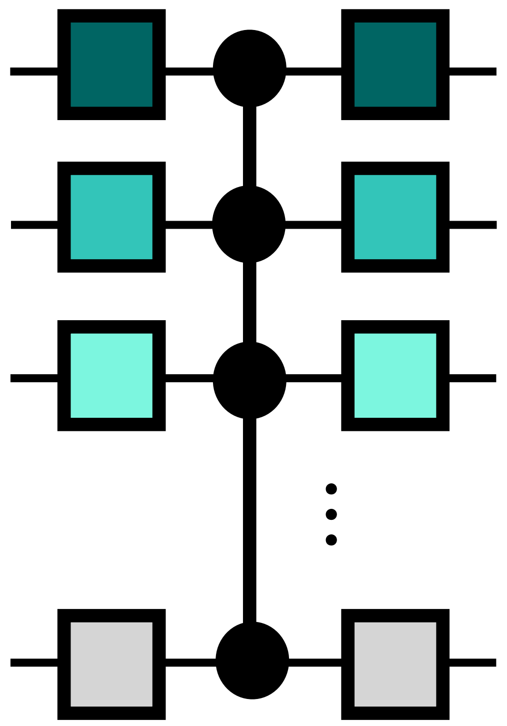

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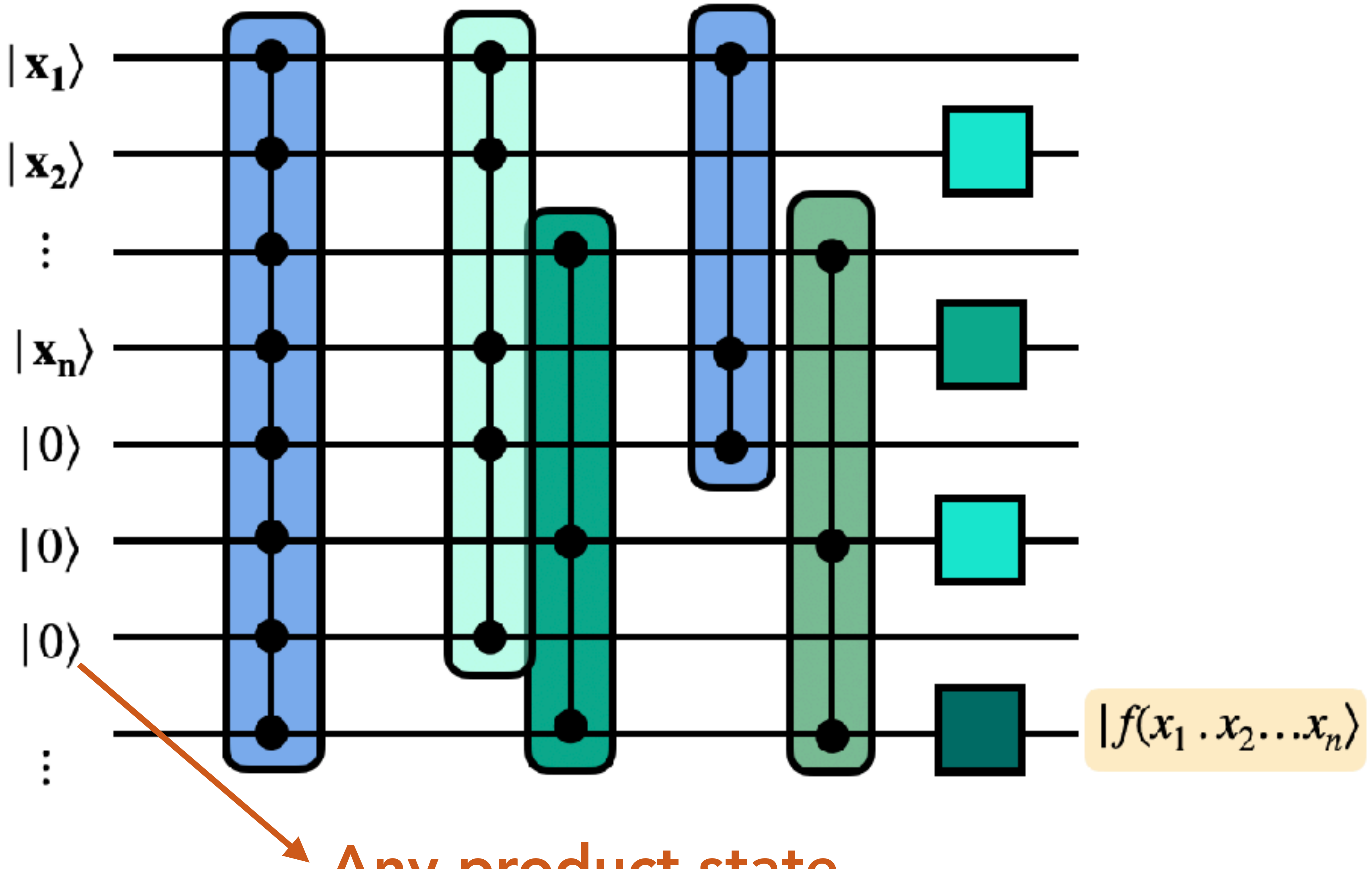


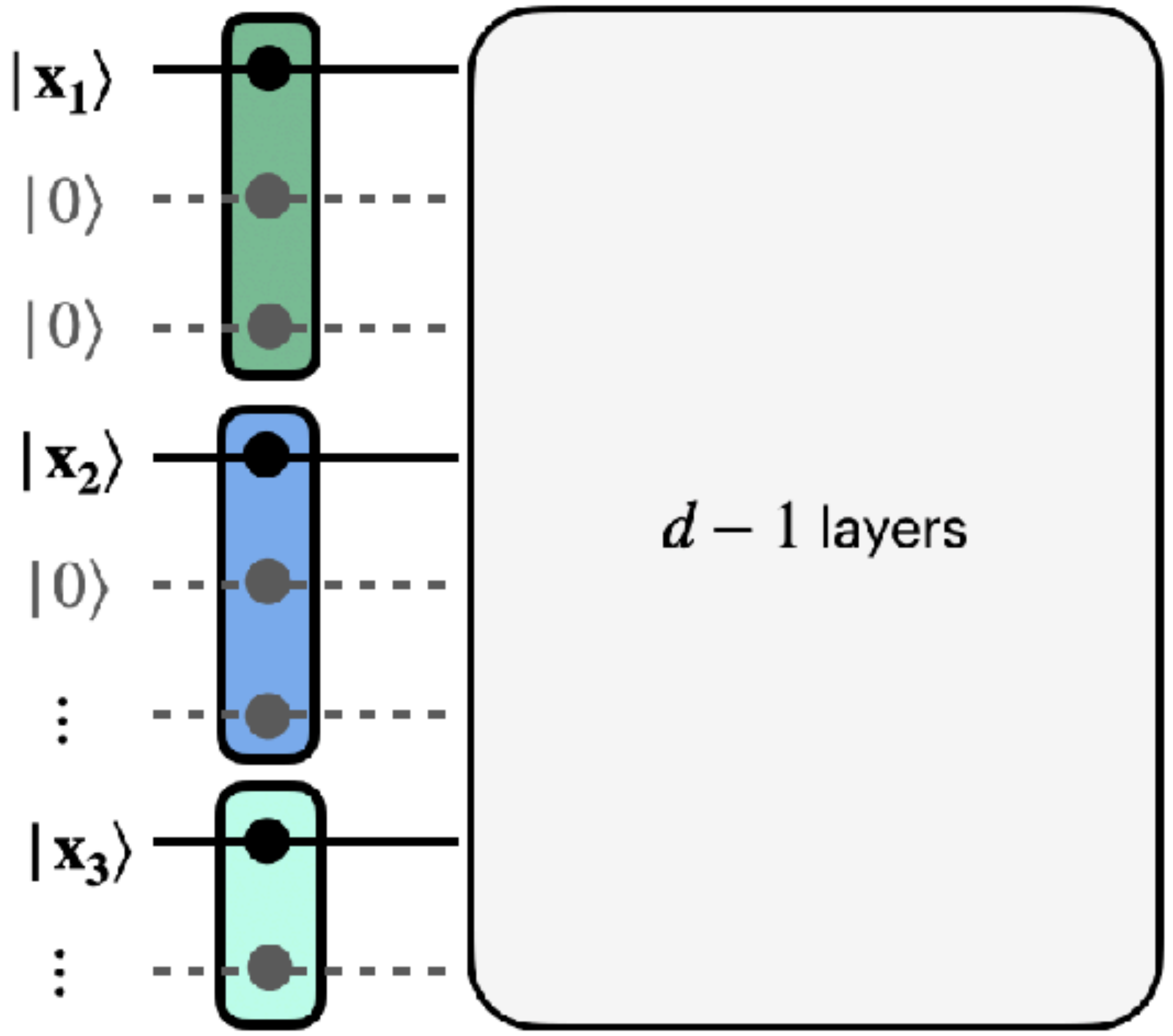
Any product state

Parity, Majority $\notin \mathrm{QAC}^0(\exp o(\sqrt{n}), 3)$

Proof Sketch:

cleaned-up-QAC^0

≤ 1 input wire per **Layer 1** gate



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1 **Clean-up lemma**

$f(x) =$ Parity/ Majority:

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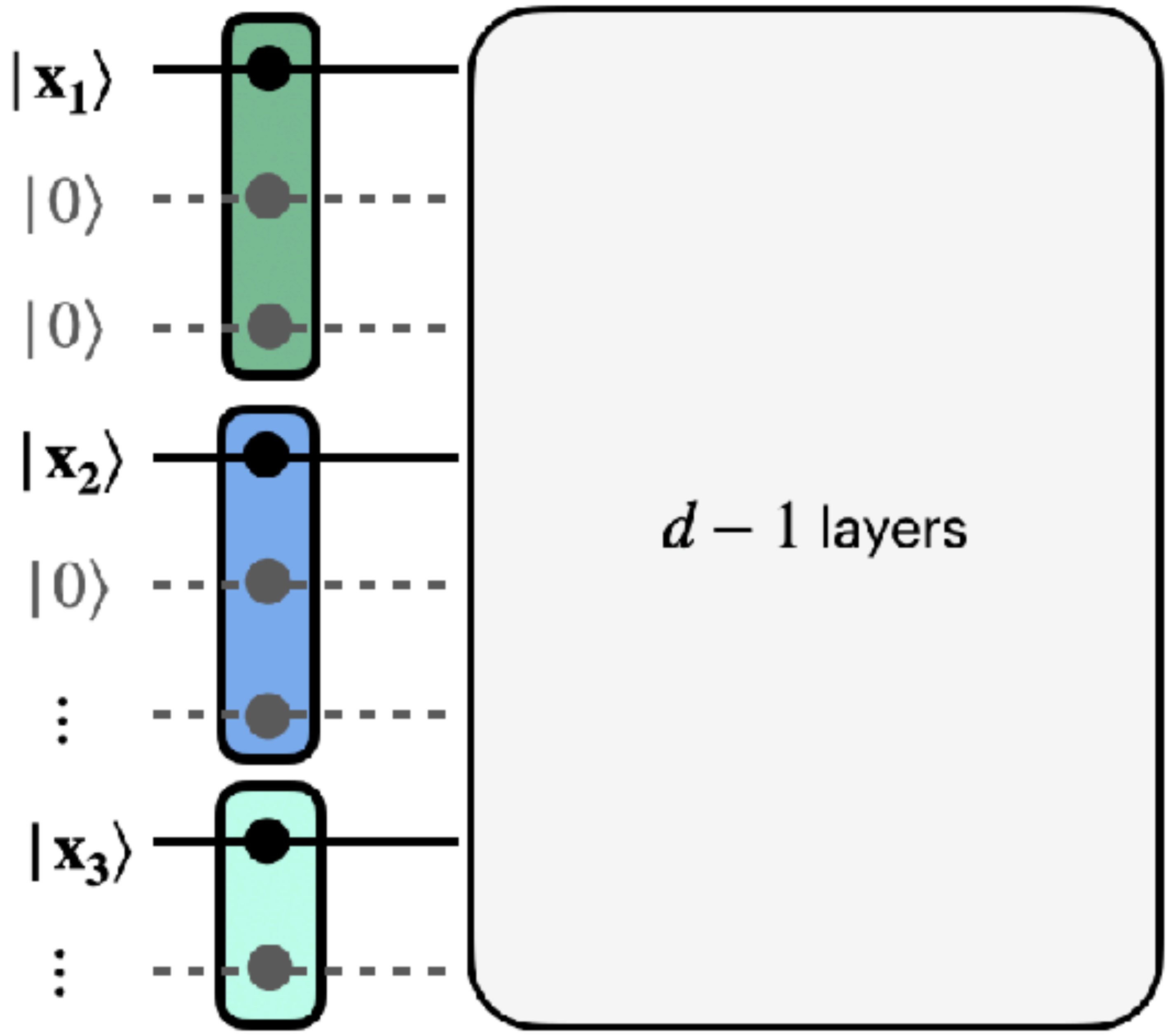

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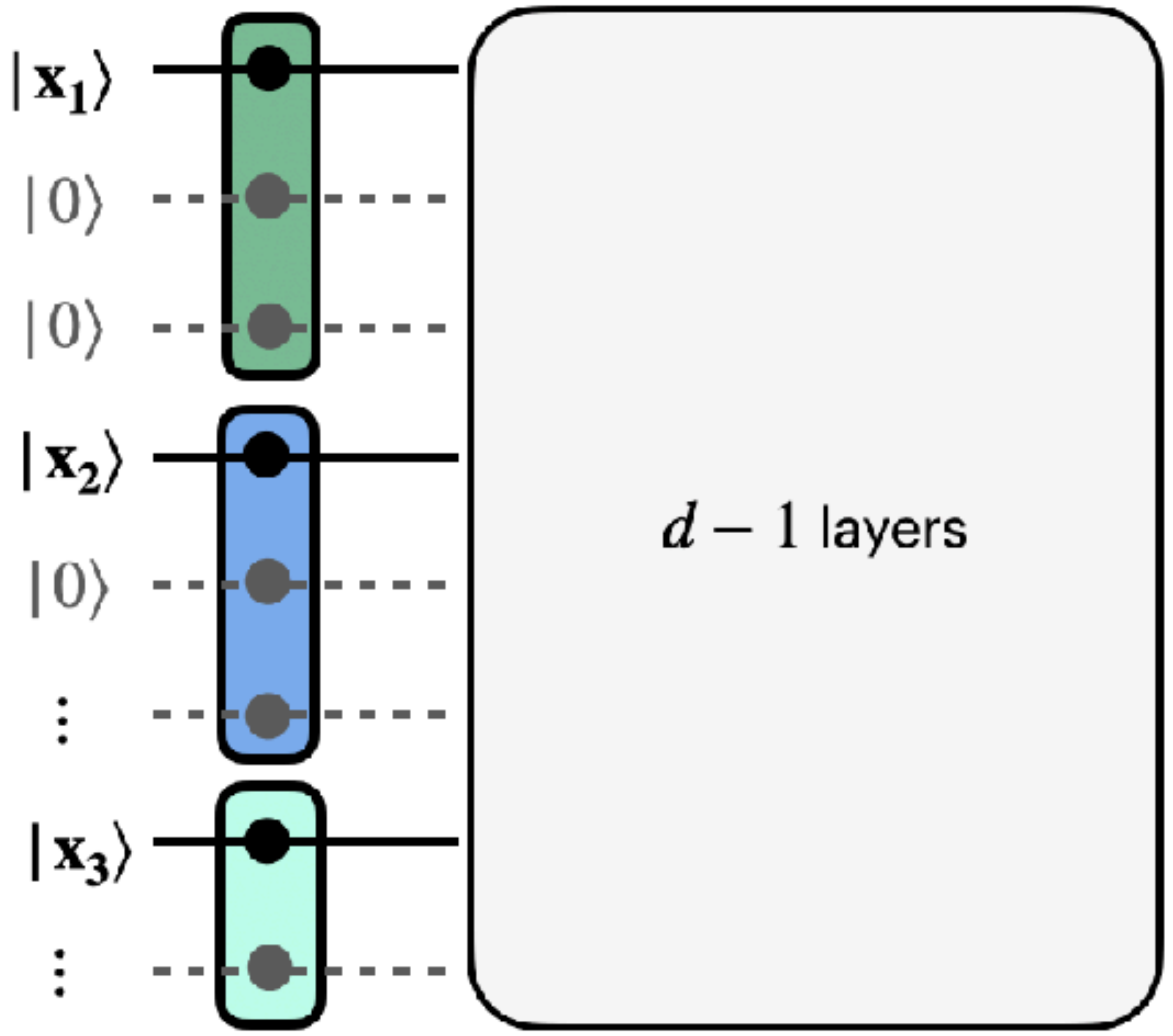

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3 [Has89] $\implies$ Parity, Majority $\notin \mathrm{AC}^0(\exp(o(\sqrt{n}), 3)$

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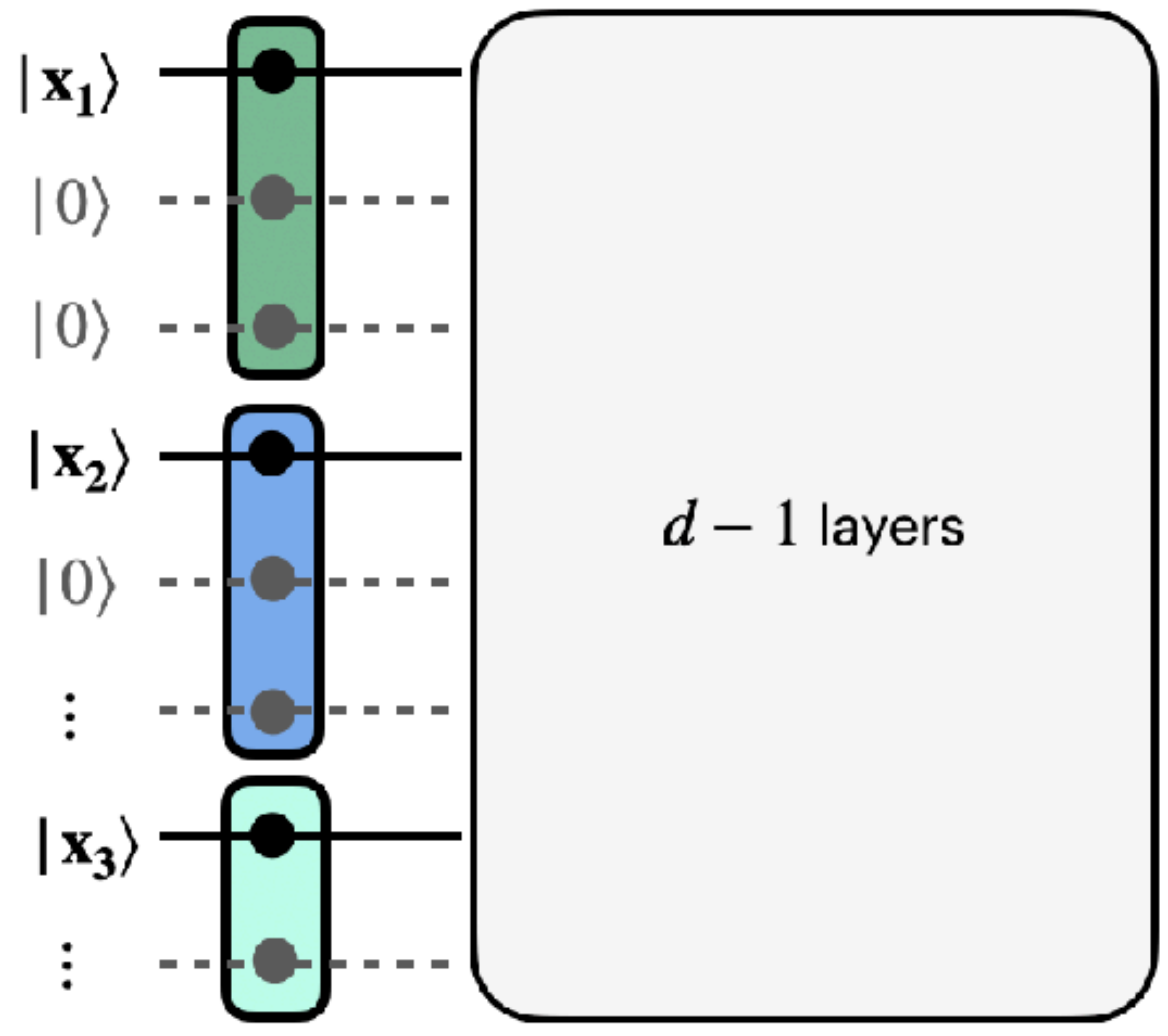

The clean up lemma

Lemma 1: Suppose $\exists\ C$ a depth-d, size-s QAC^0 circuit C computing **Parity** of n-inputs.

Then, $\exists$ a depth-d, size-s, **cleaned-up-QAC^0** circuit C' for computing **Parity** of $n' \geq n/3$ inputs.

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The clean up lemma

also Majority

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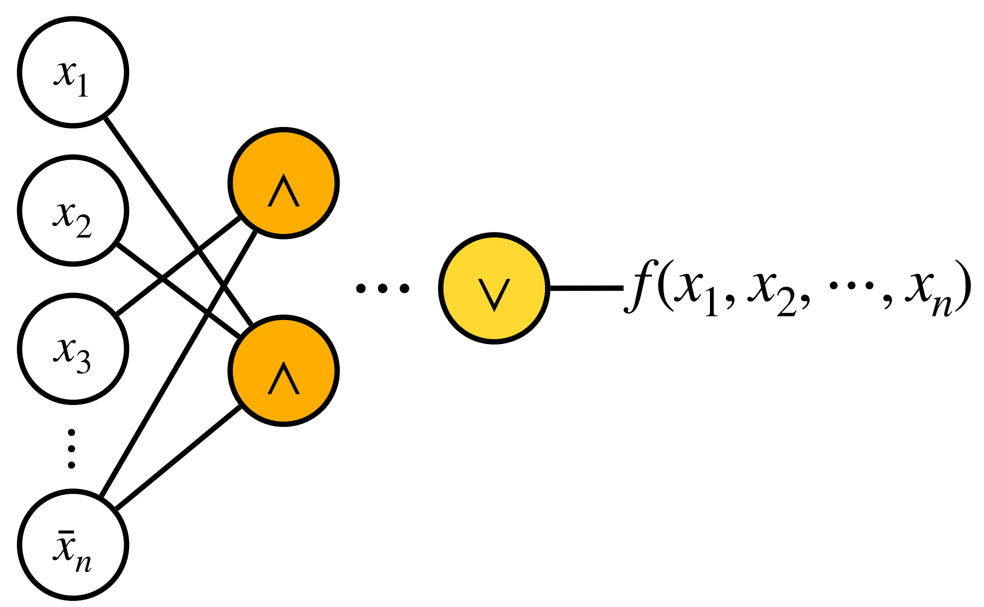

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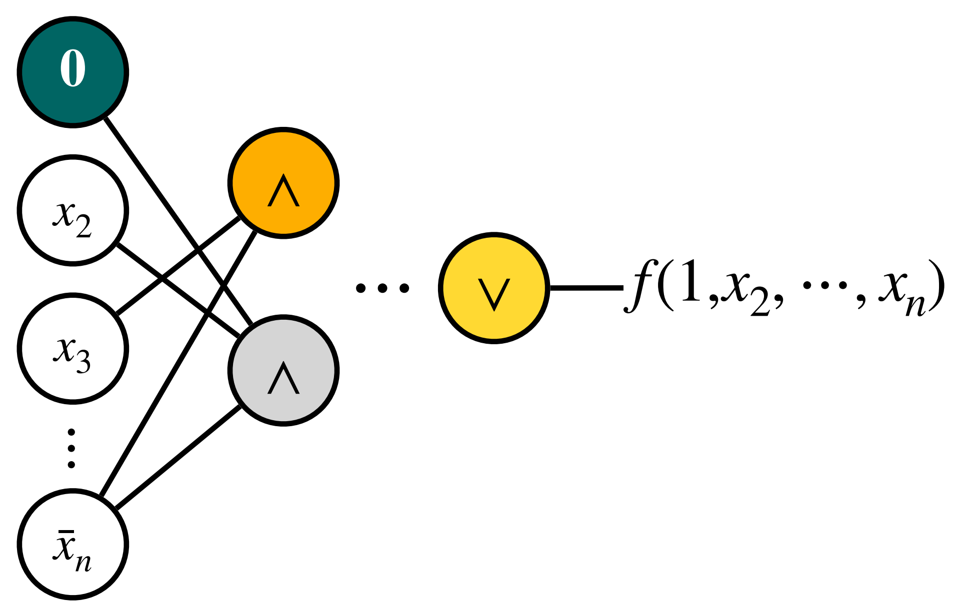

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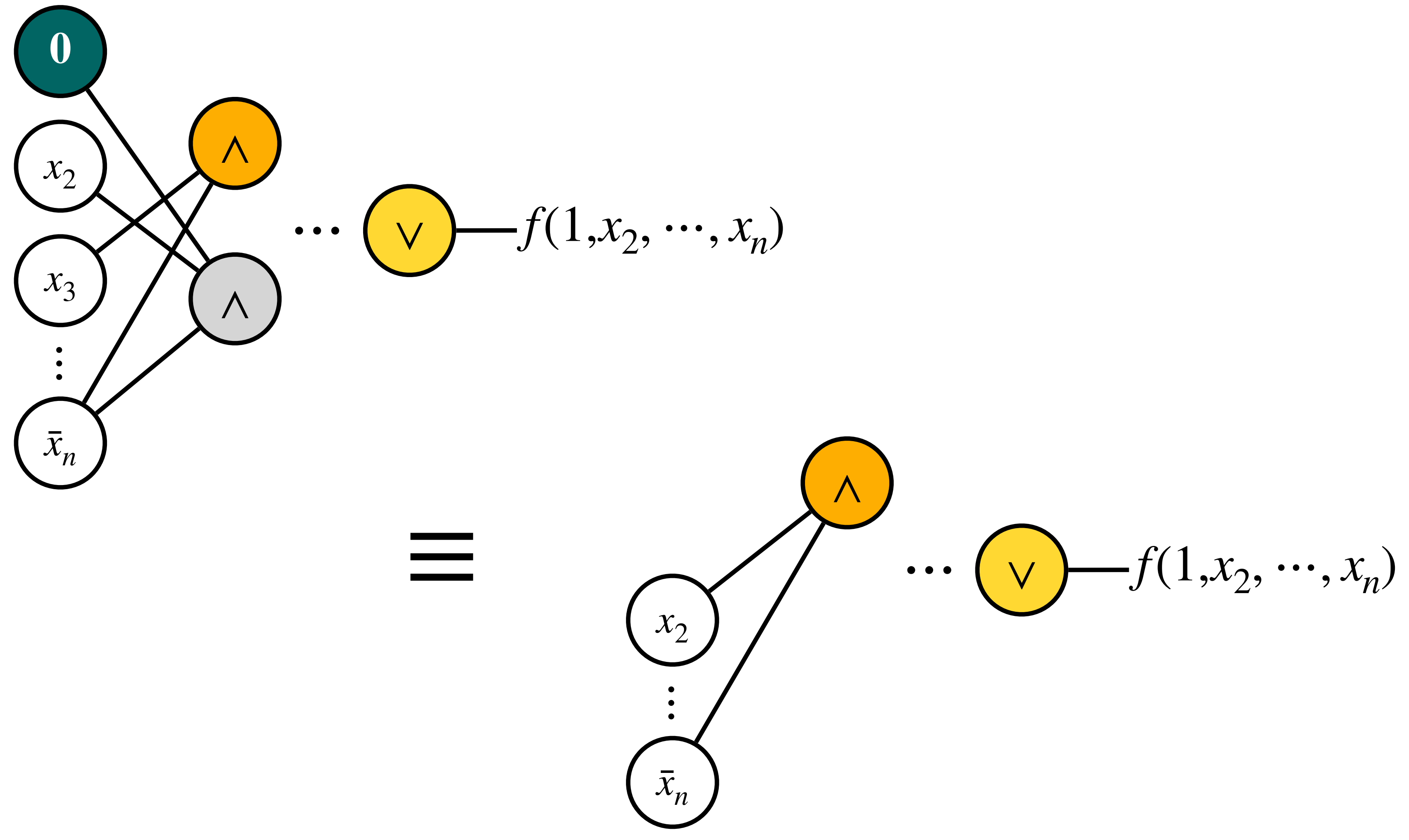

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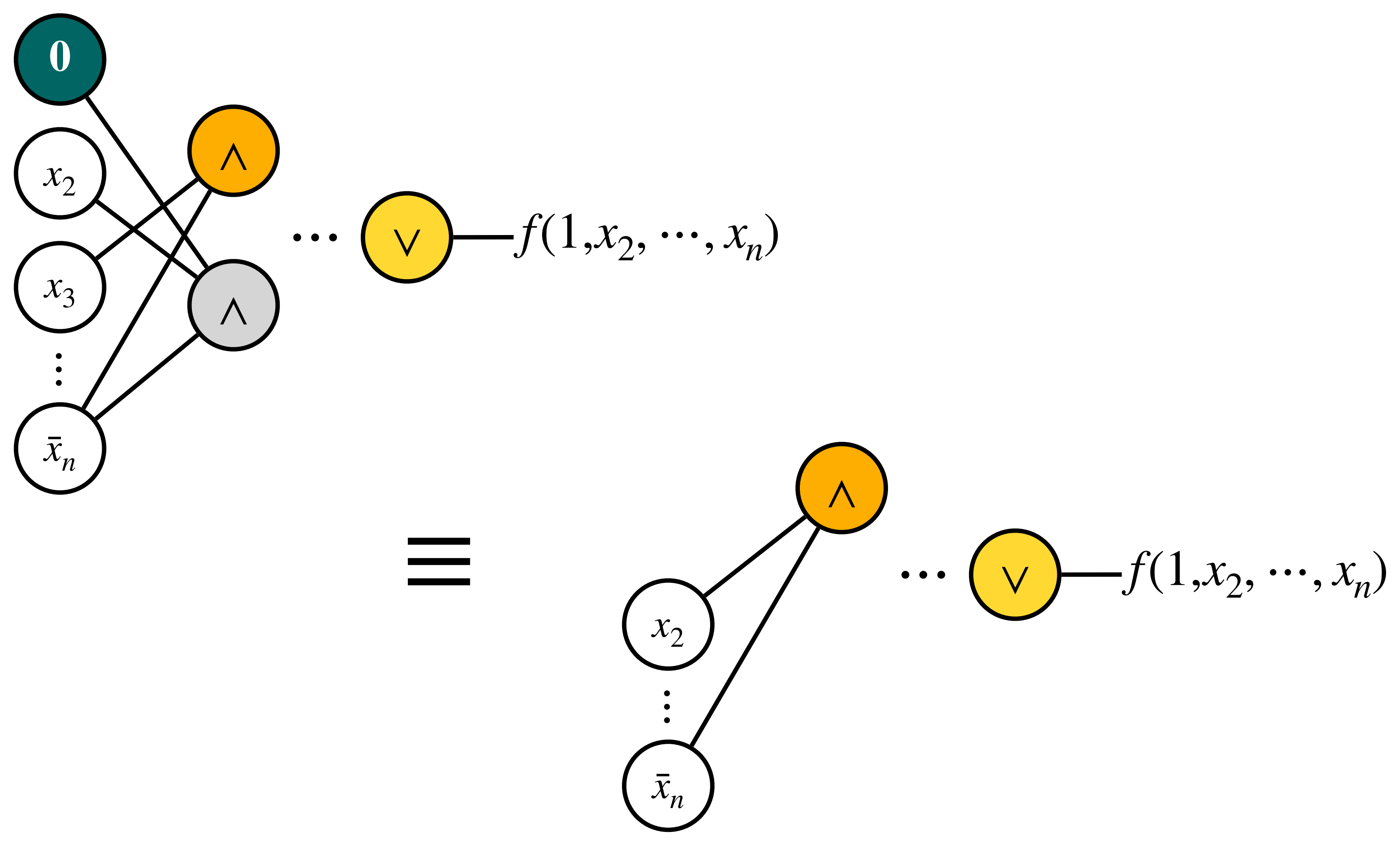

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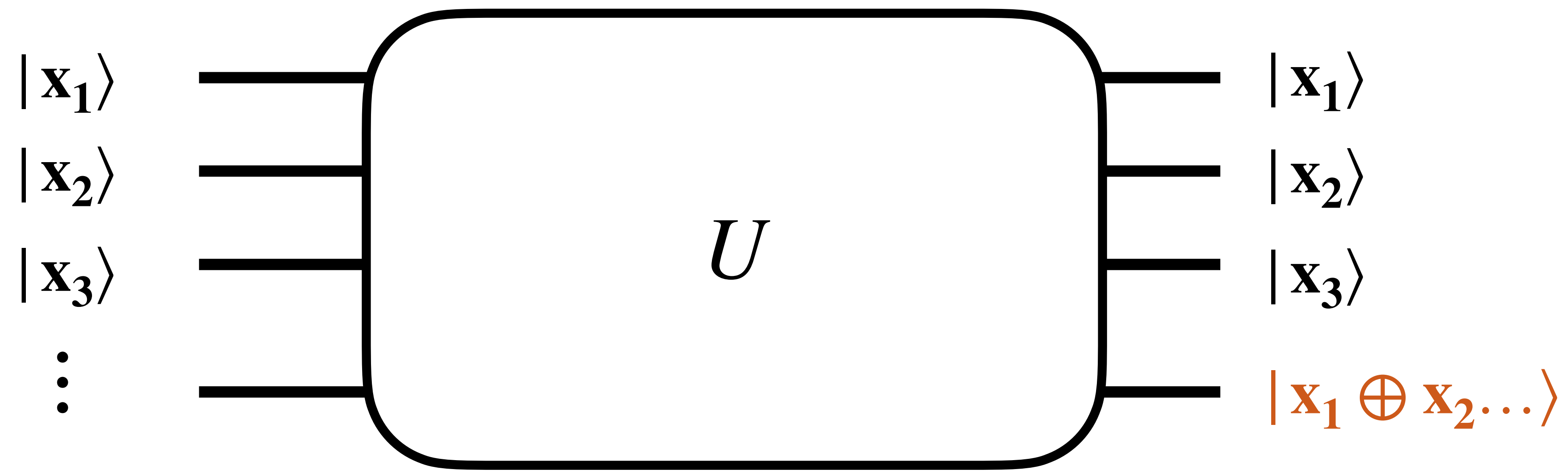

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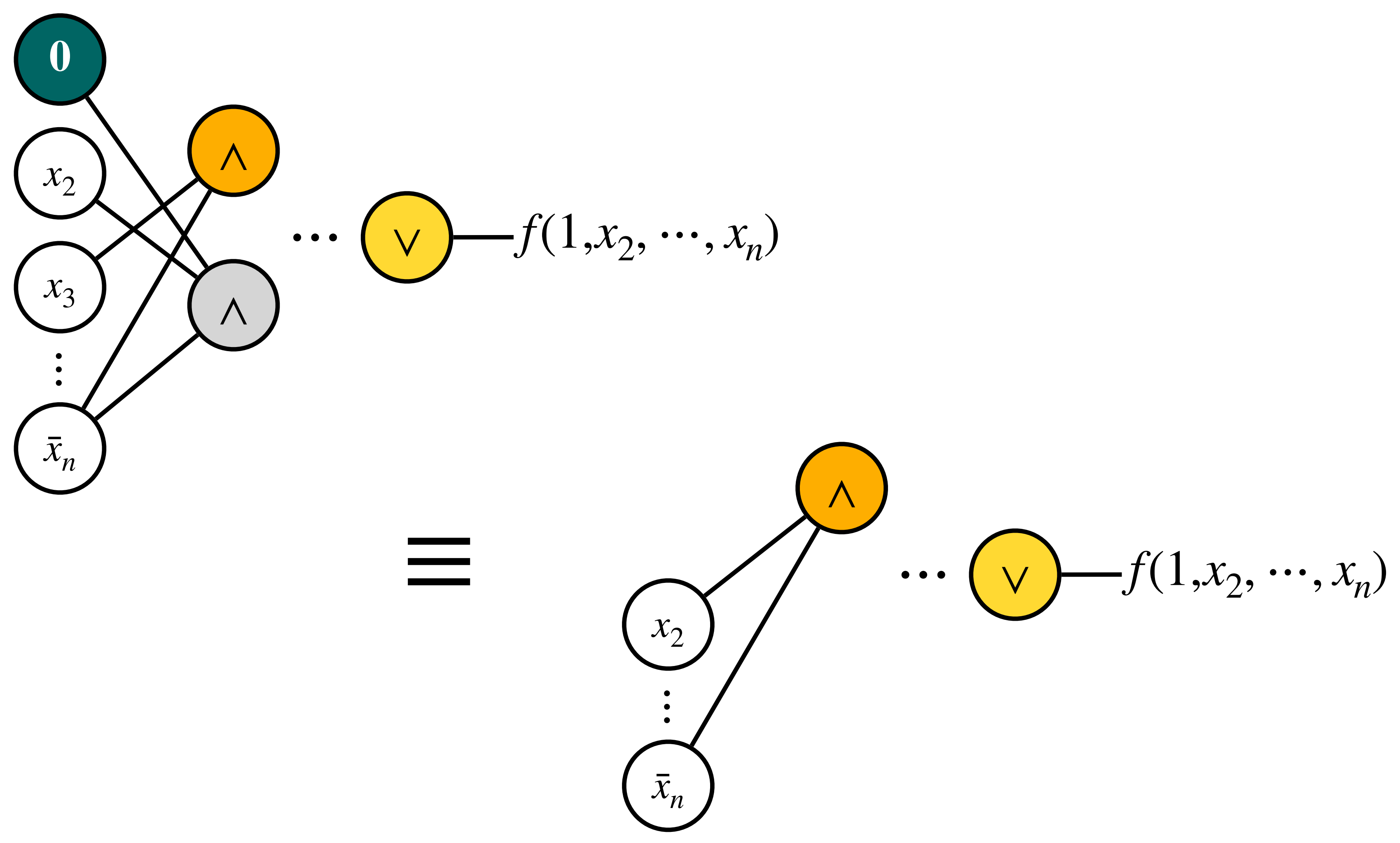

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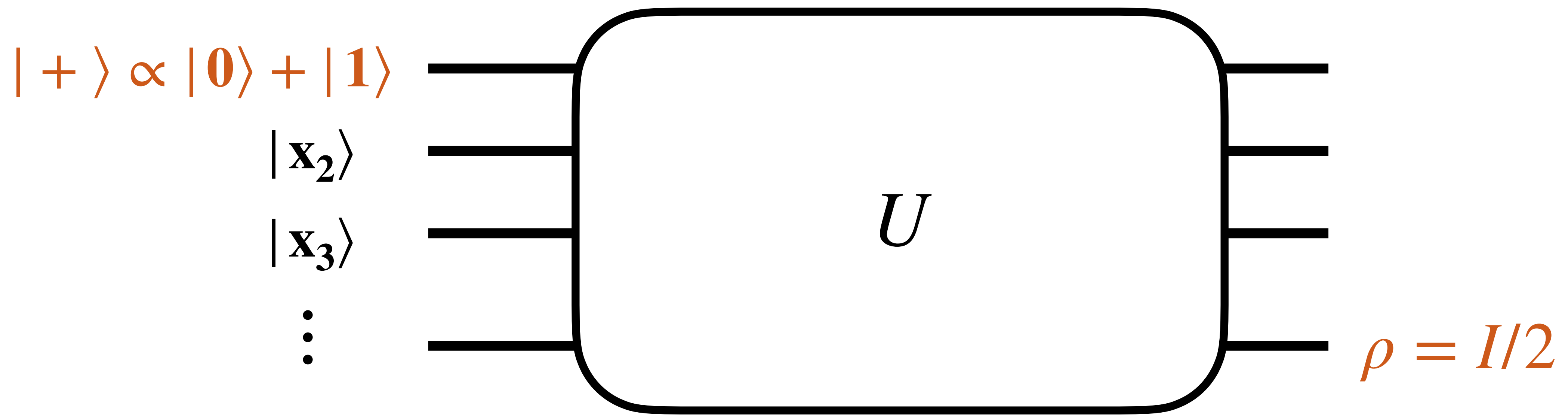

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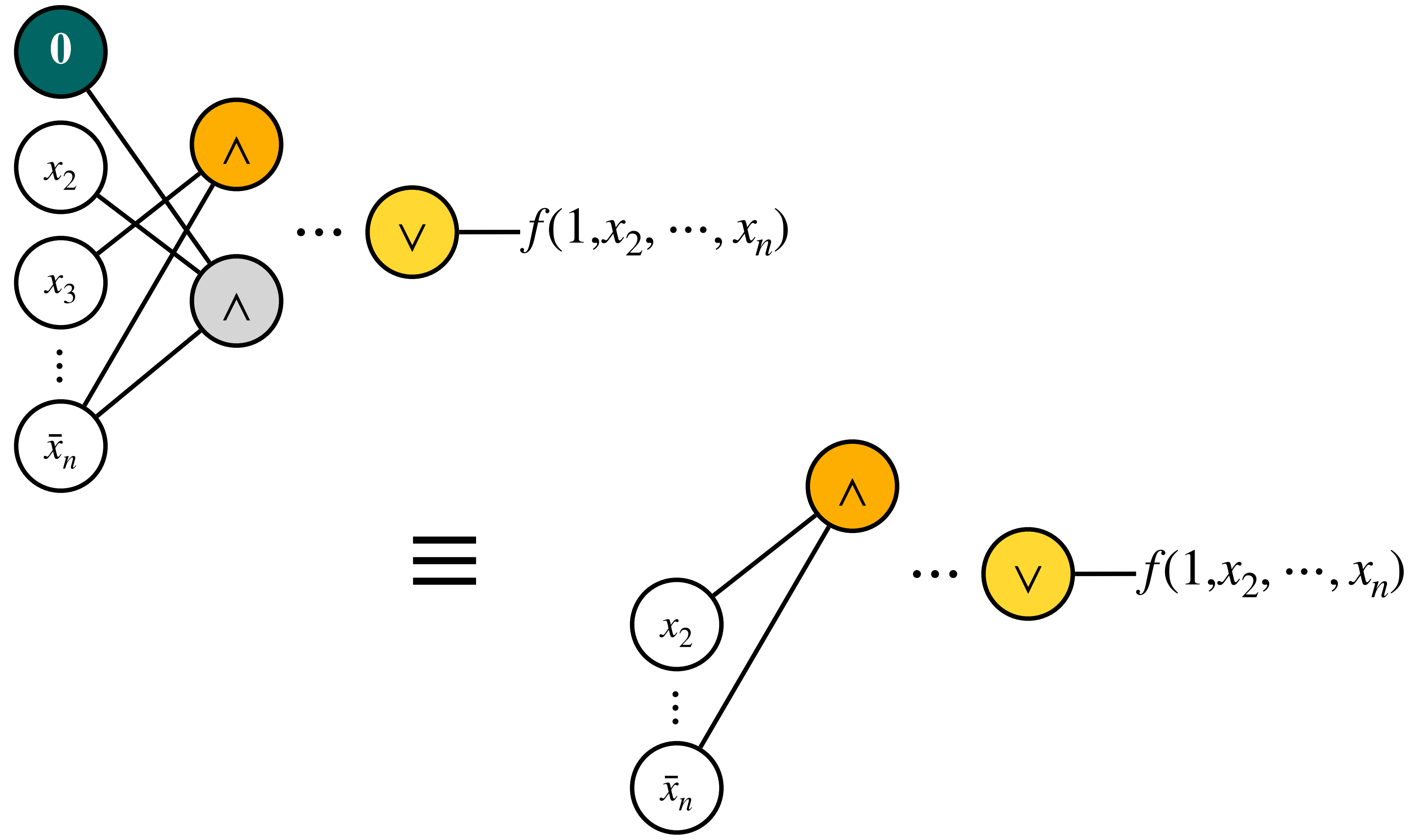

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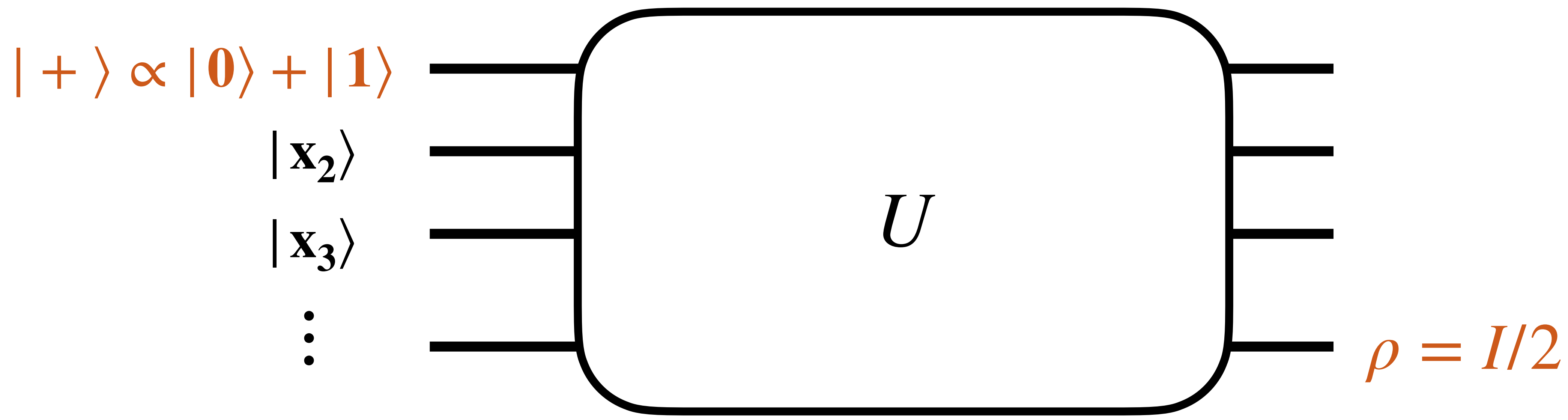

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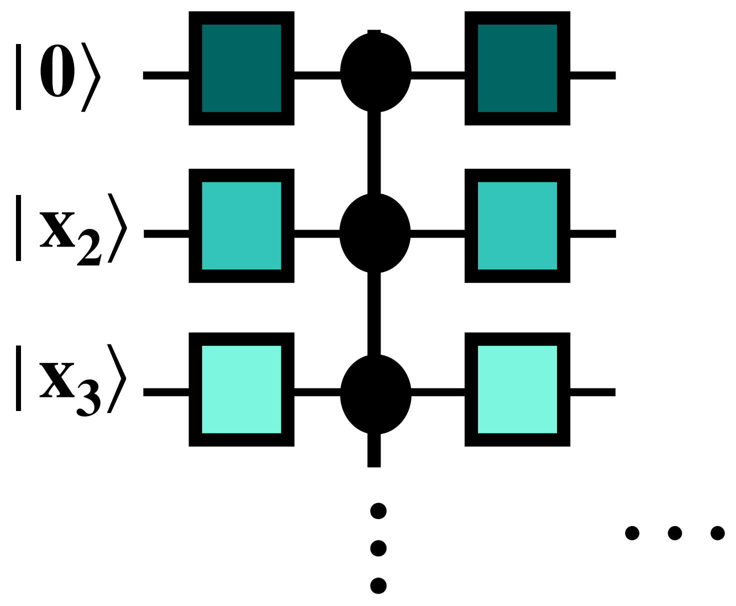

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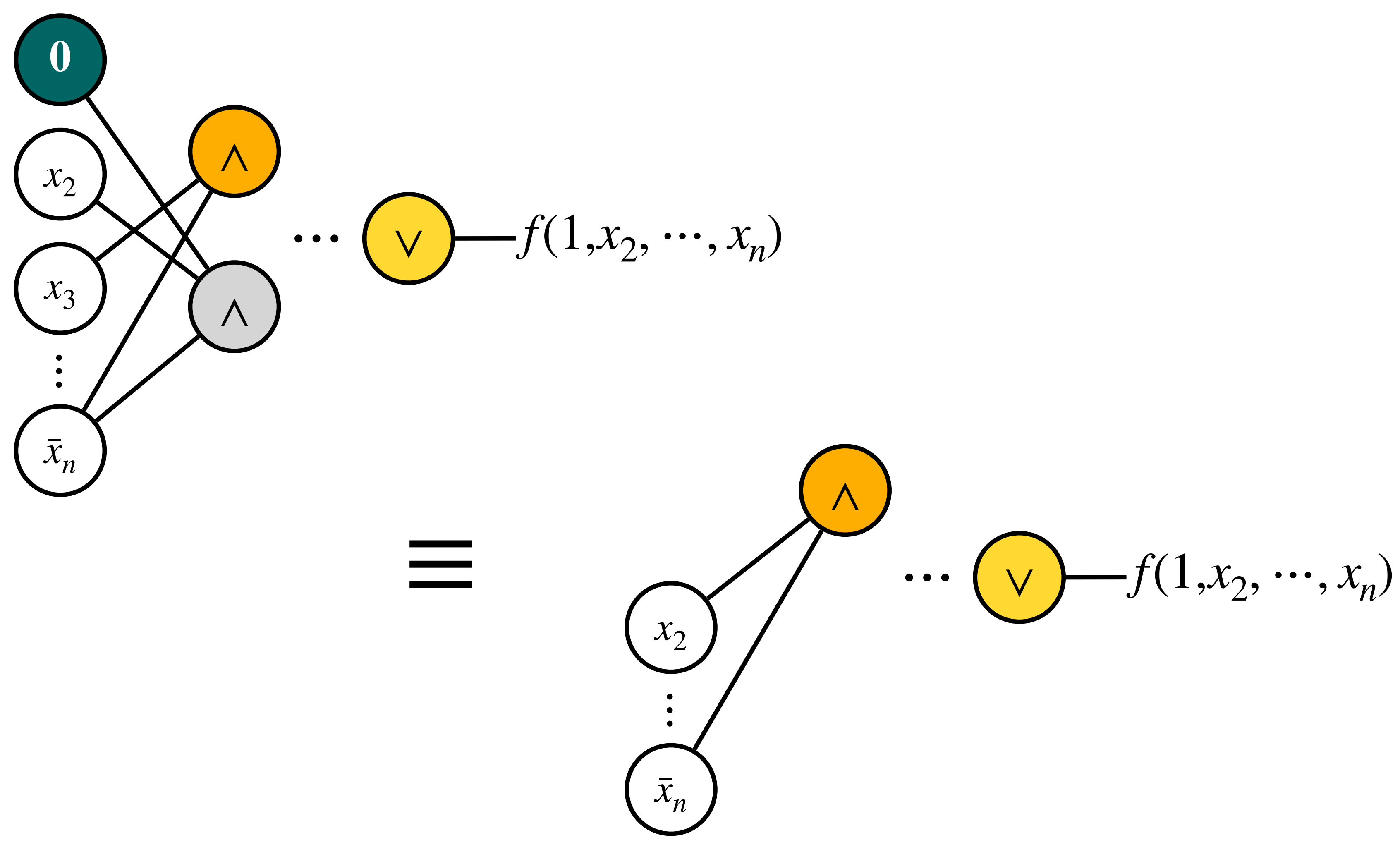

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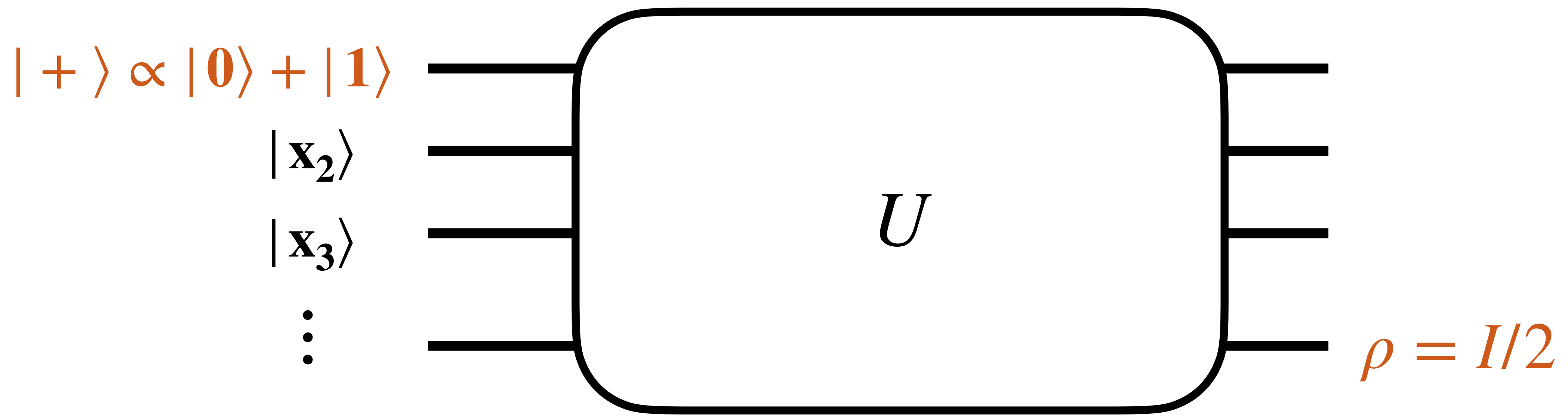

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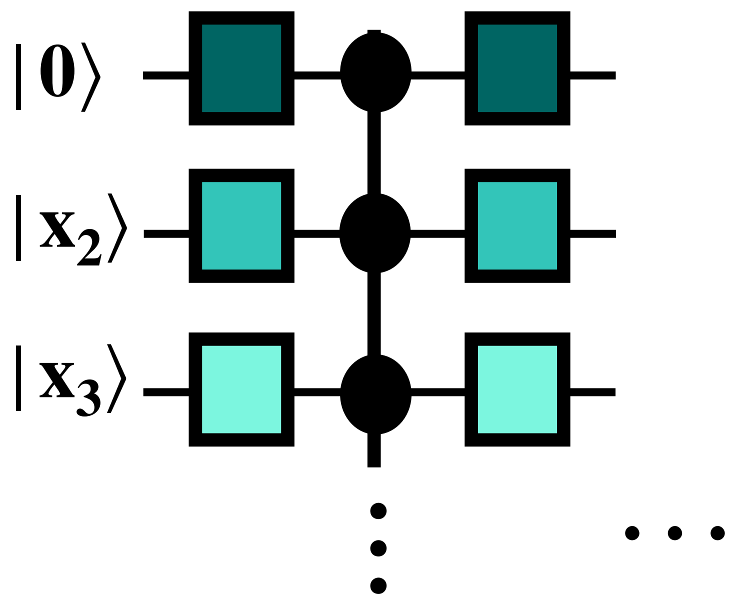

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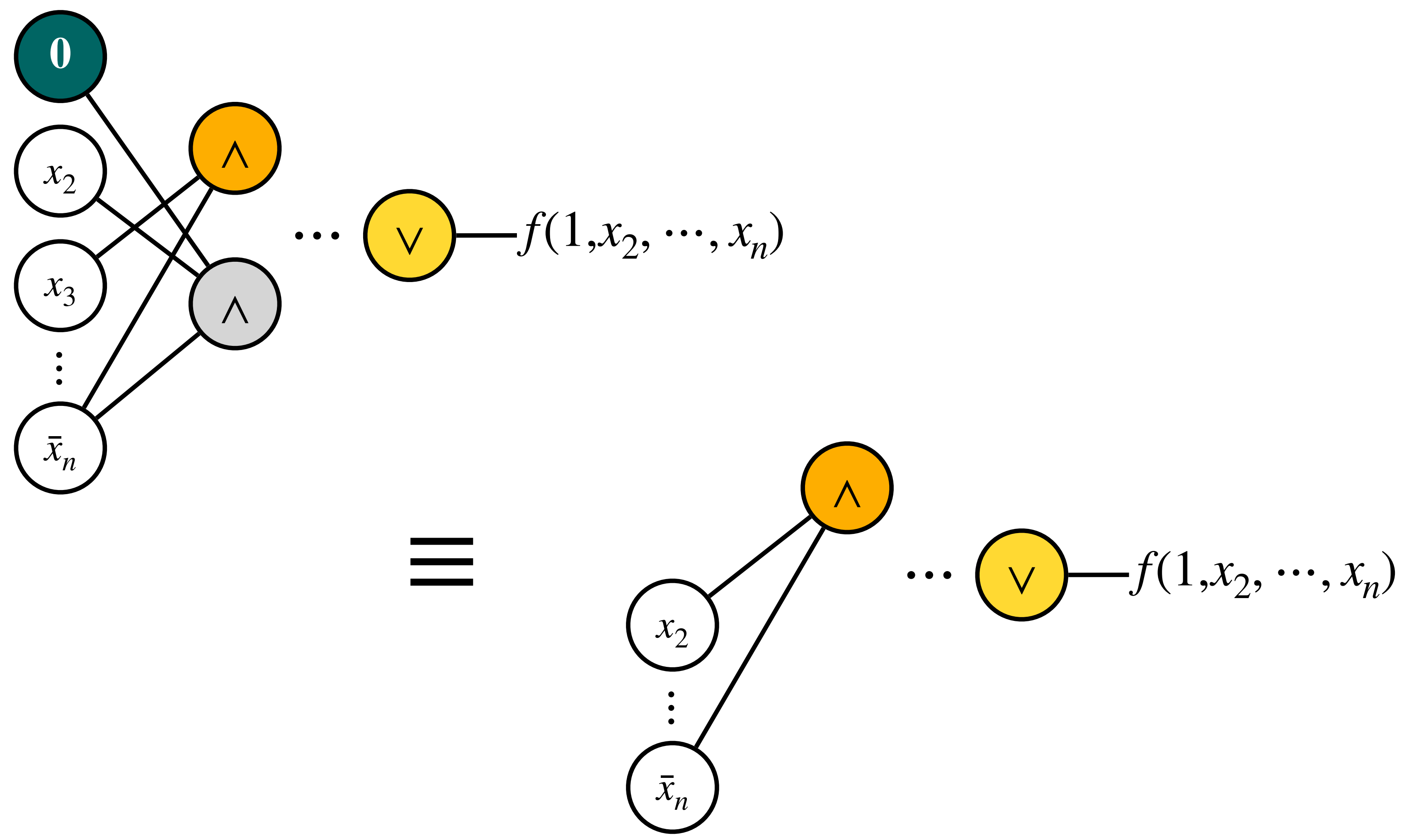

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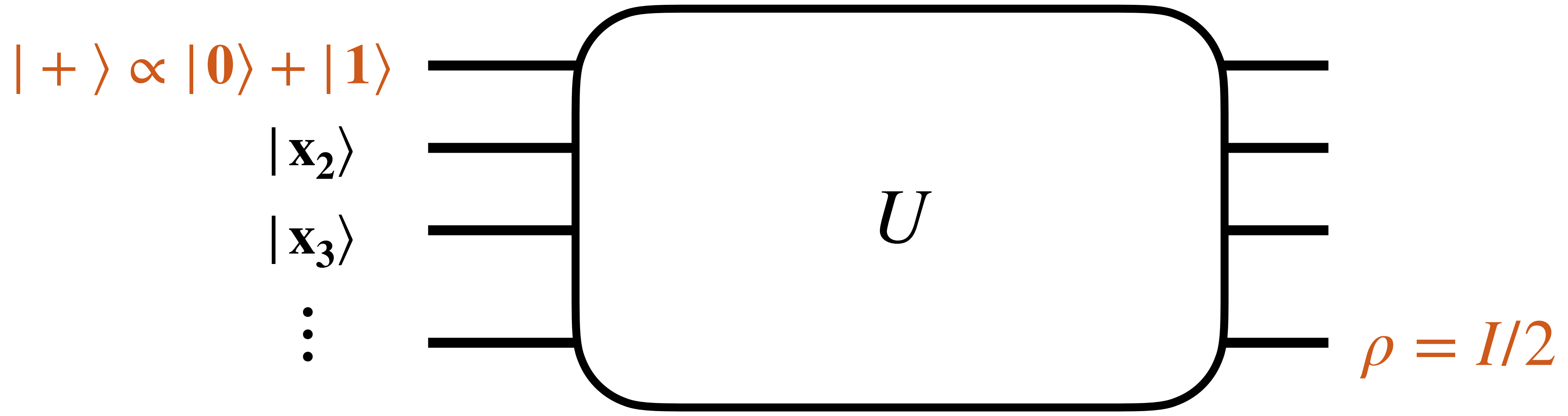

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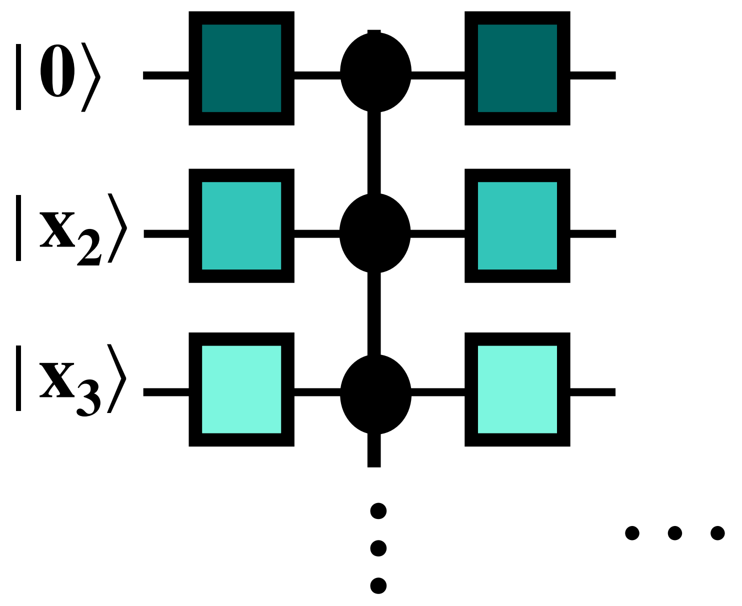

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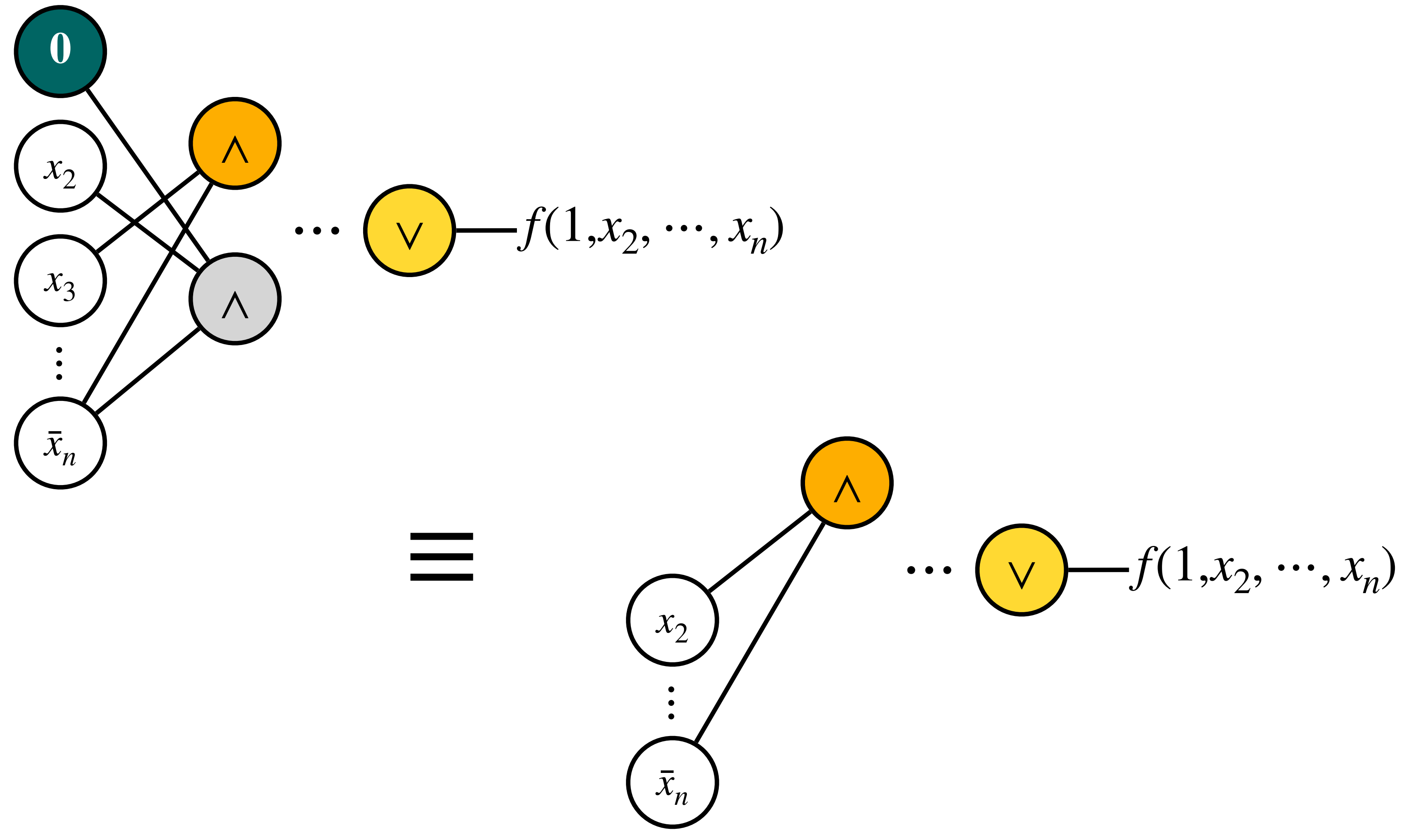

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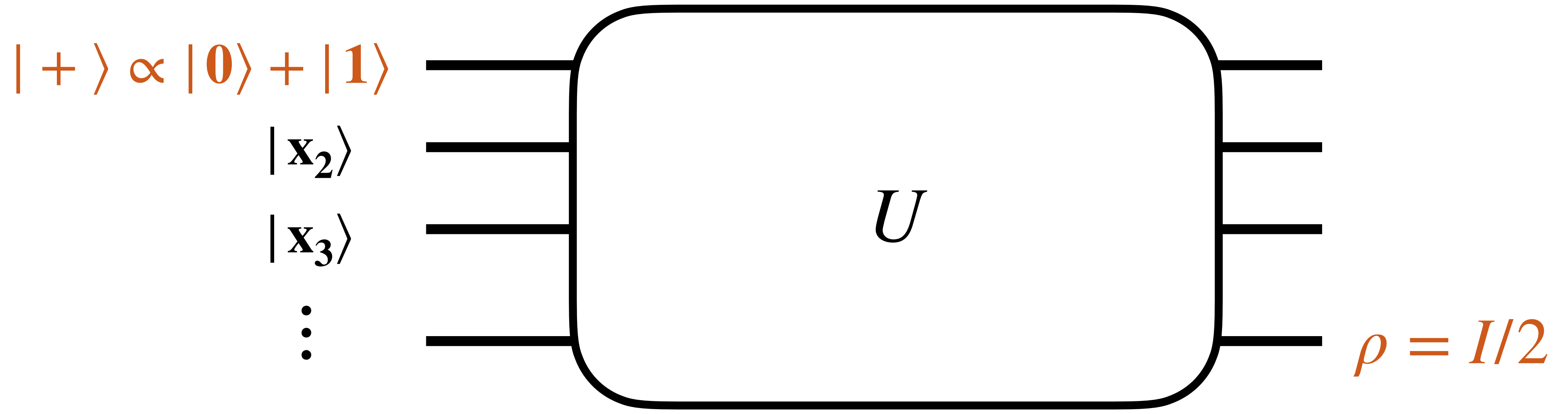

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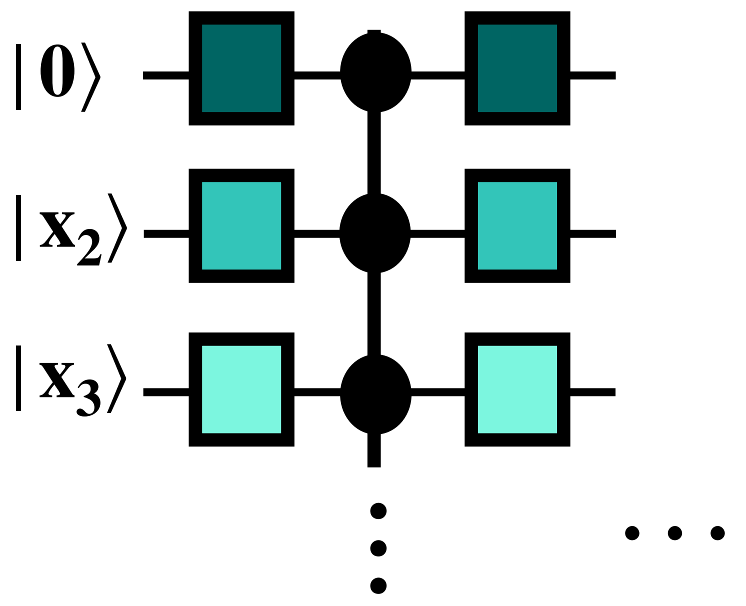

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Subspace orthogonal to $|\vec{\theta}\rangle_S$?

Restrict to *two qubit state* to preserve parity and simplify circuit!

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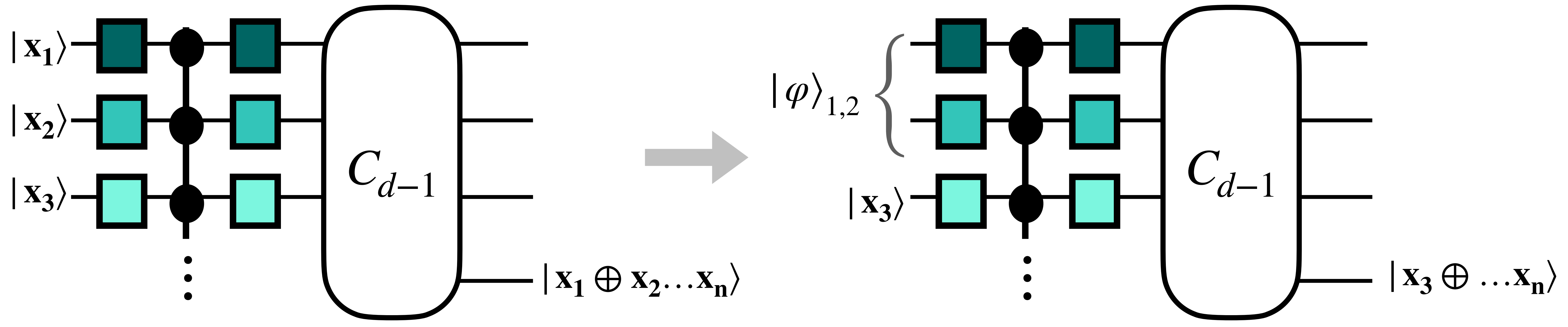

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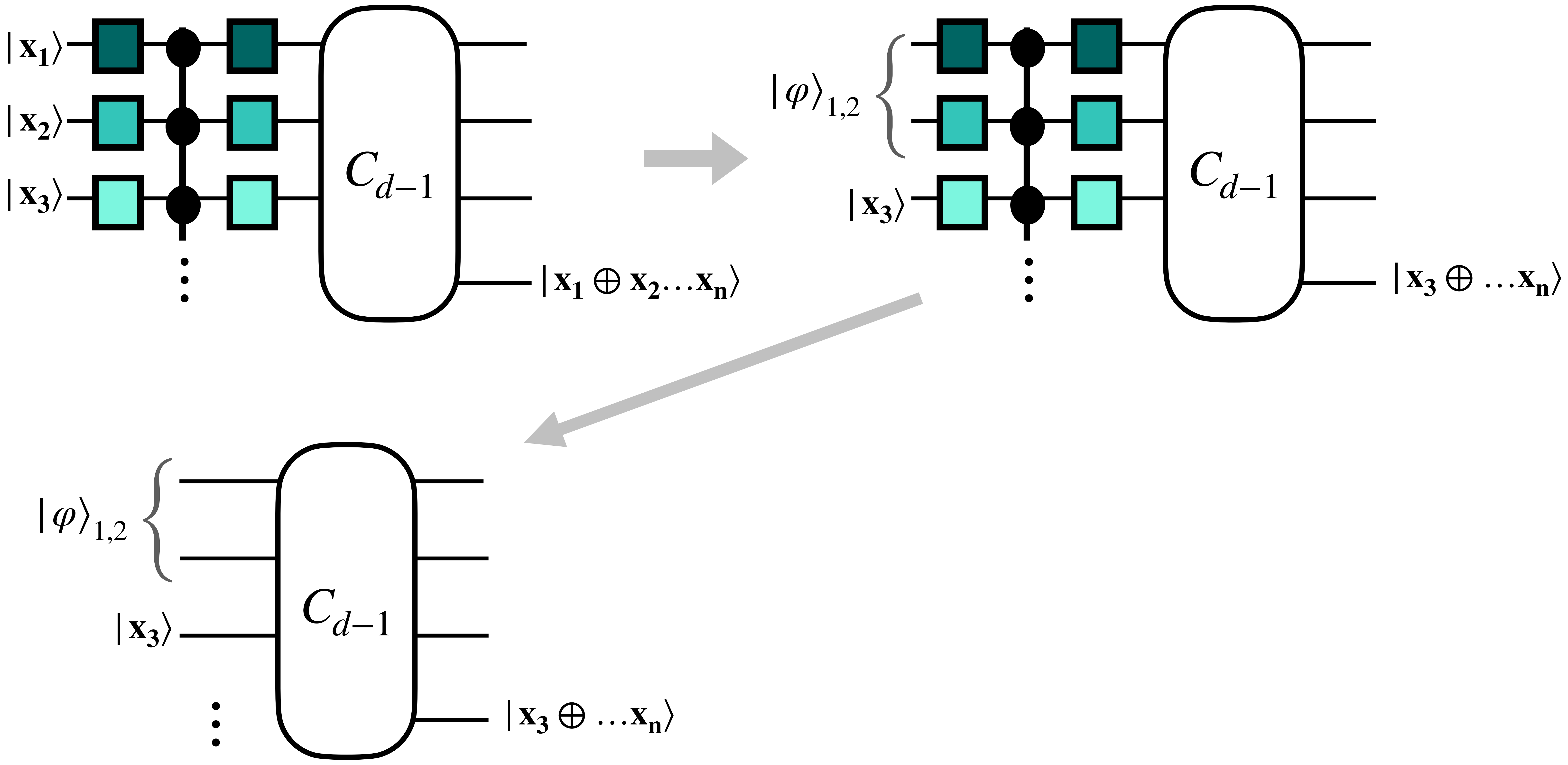

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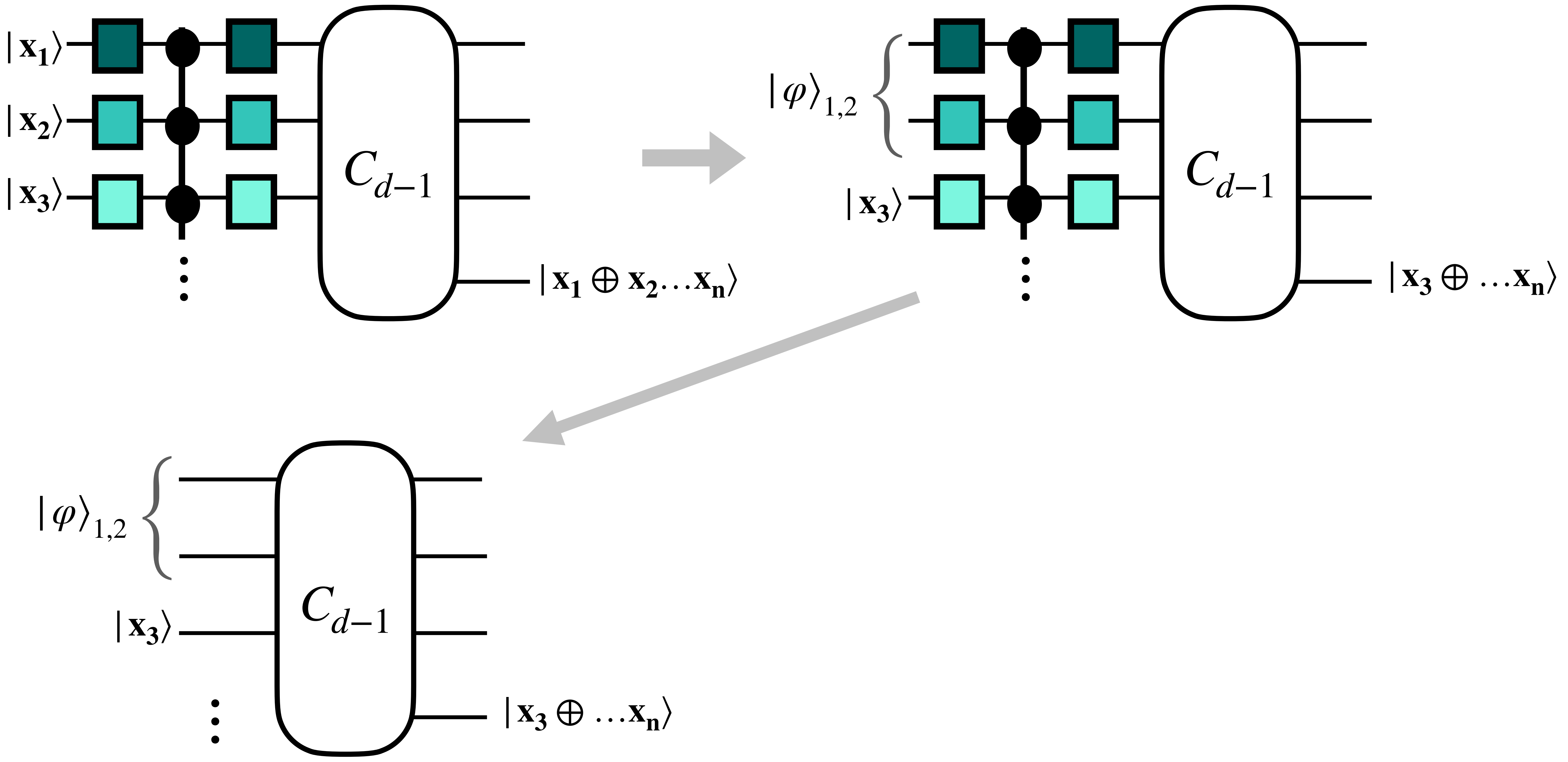

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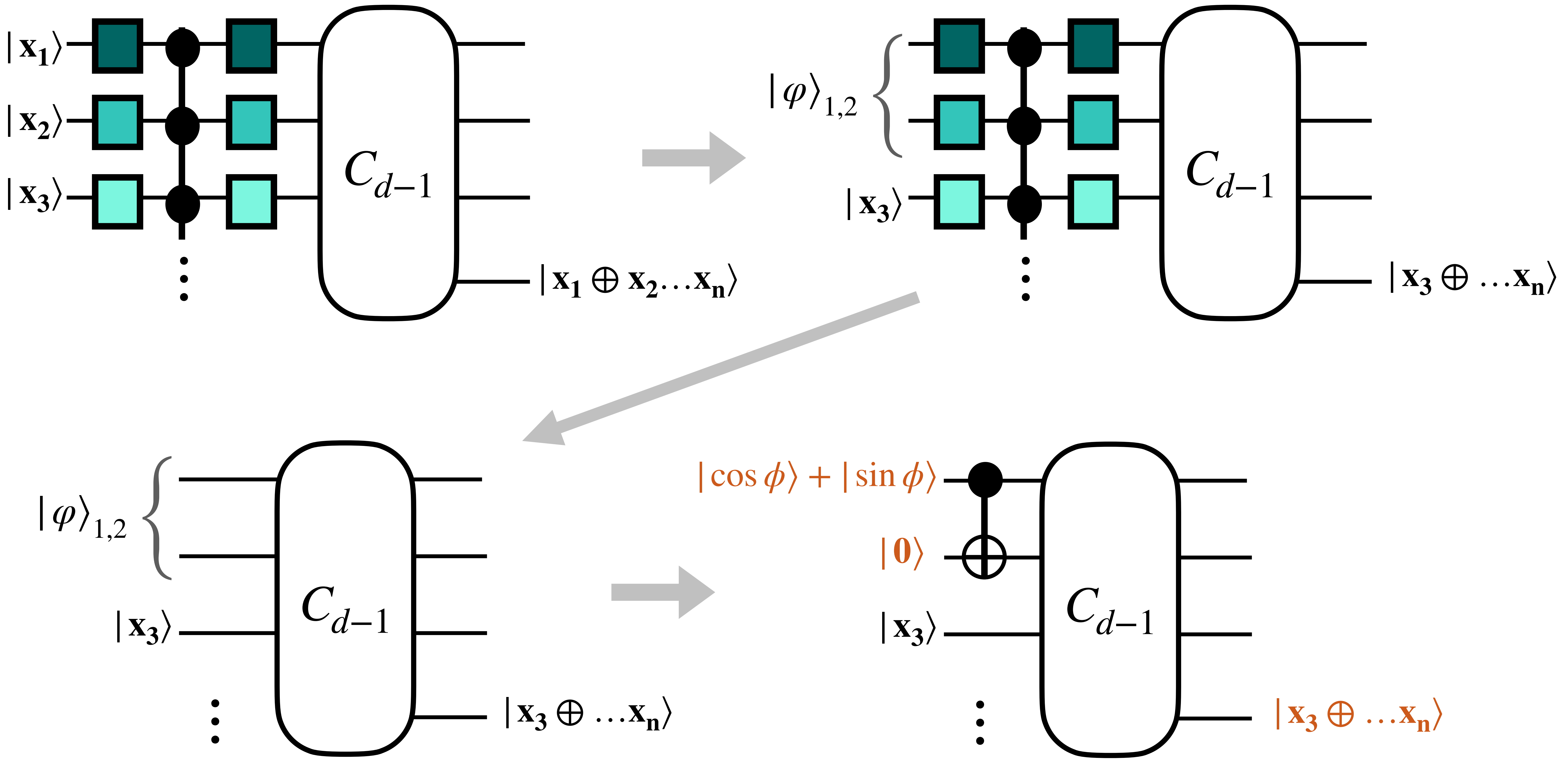

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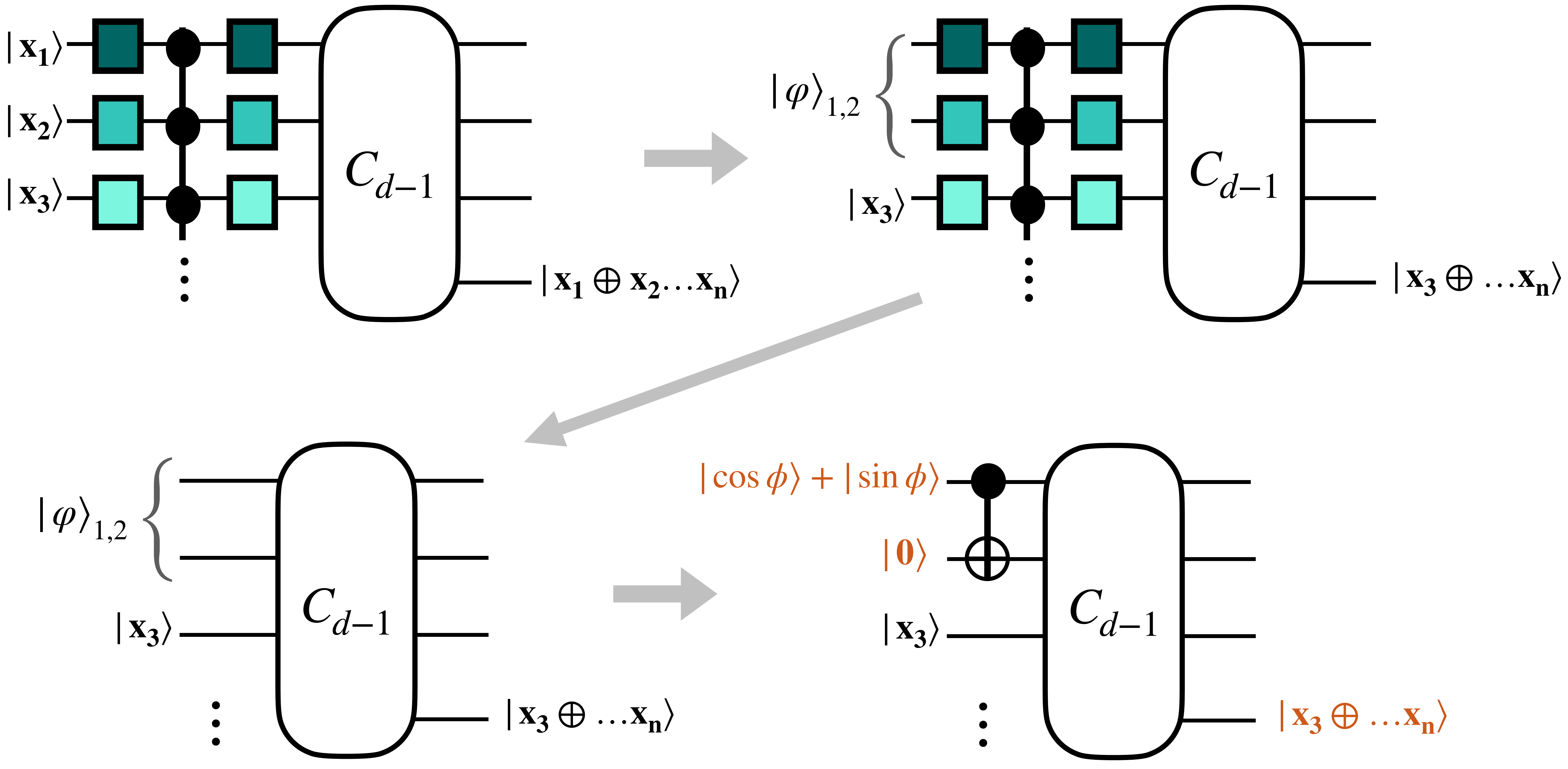

Not a product state...

Rest of the proof

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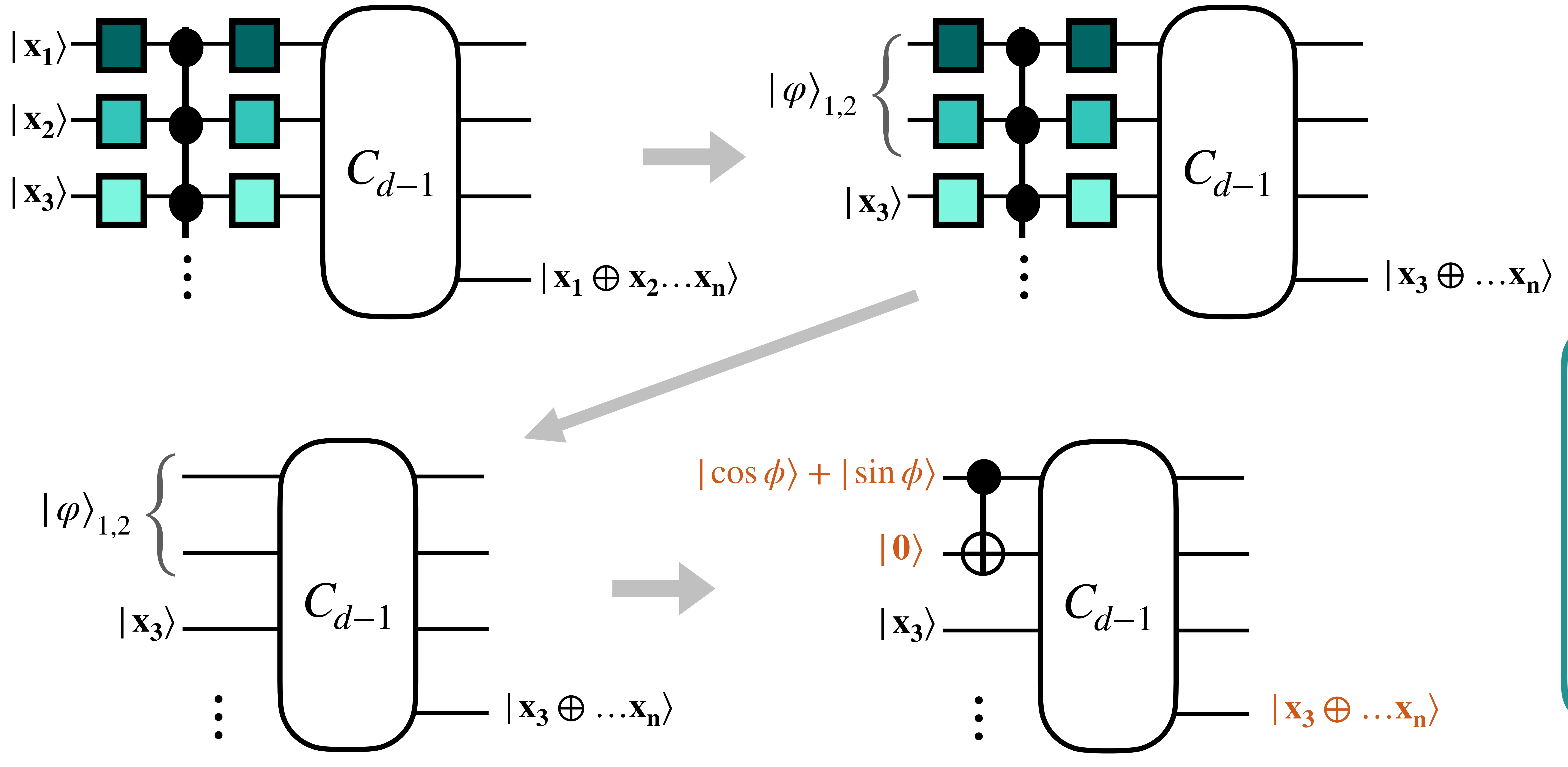

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- 2 input wires $\rightarrow$ restrict one to $|\mathbf{0}\rangle$

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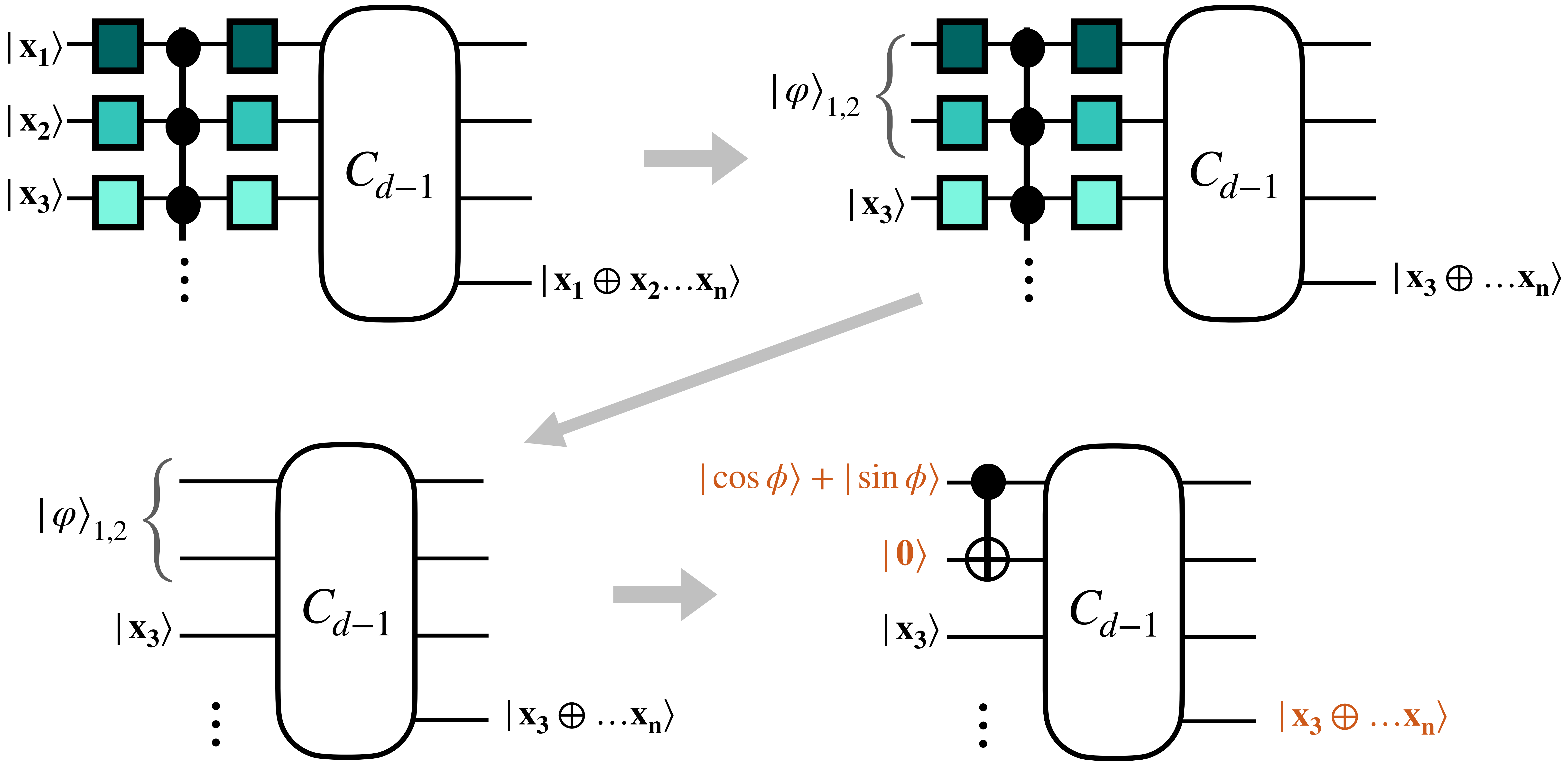

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Rest of the proof

- 2 input wires → restrict one to $|\mathbf{0}\rangle$
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Proof of Clean-Up: Quantum Restrictions

- Parity invariant : Any vector $\in \text{span}\{|\mathbf{00}\rangle, |\mathbf{11}\rangle\}$
- Kill $U_S = I - 2|\vec{\theta}\rangle\langle\vec{\theta}|_S$: Orthogonal to $|\theta_1\rangle \otimes |\theta_2\rangle$

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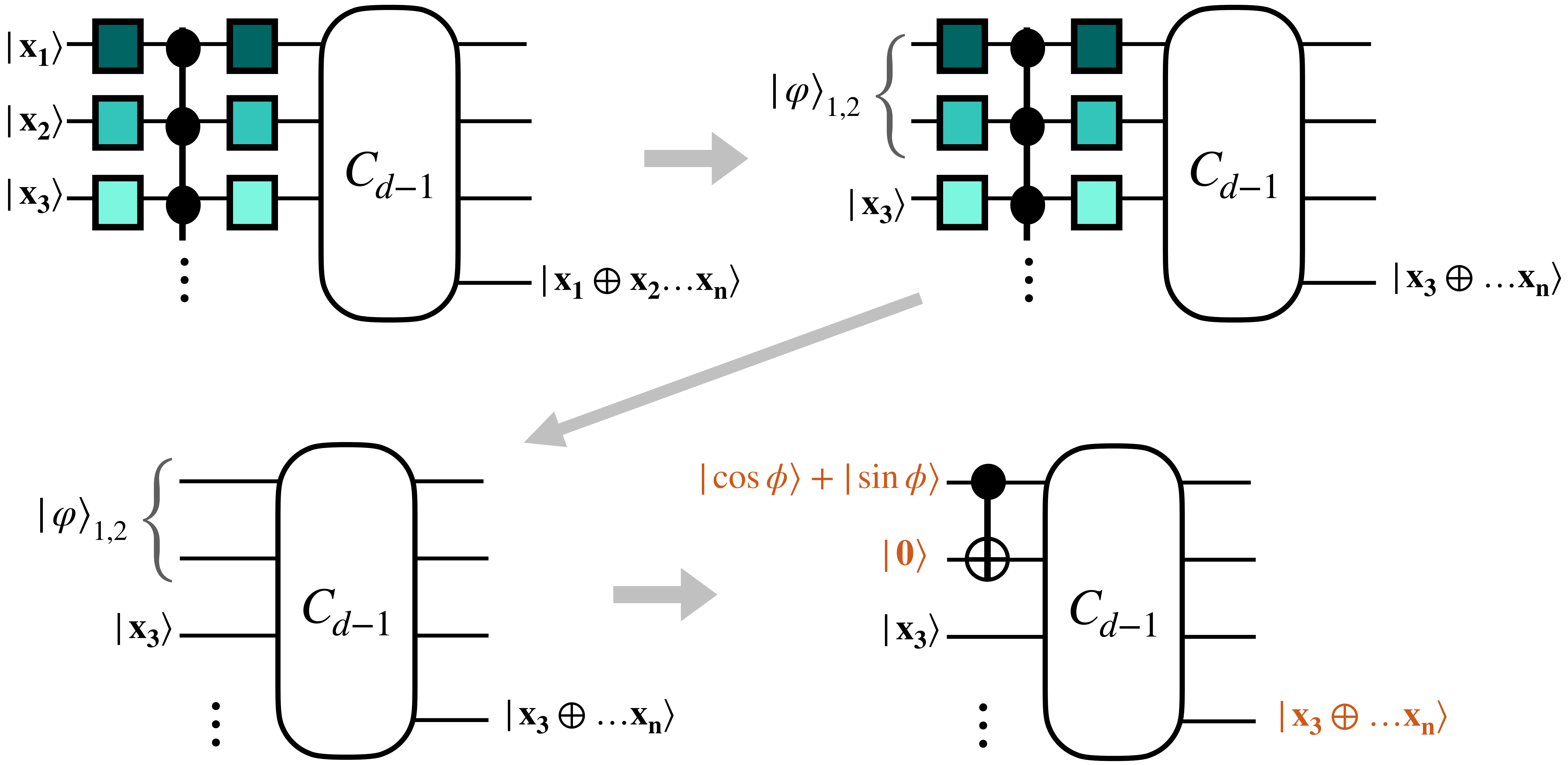

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$\Rightarrow \geq n/3$ inputs preserved
depth $\leq d$, size $\leq s$ ∎

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$\implies$ **PARITY** $\in \mathrm{QAC}^0$ [GGJ26]

Thank you